

# Eilenberg Centenary Conference

Functoriality in Surgery:  
The Geometry and some implications

## Classical Surgery

After Kervaire, Milnor, Browder, Wall and as formulated by Sullivan.

**Definition:**  $S^{\text{cat}}(M) =$

$\{(W, f) \mid W \text{ is a Cat-manifold, } f: W \rightarrow M \text{ is a simple homotopy equivalence}\}$   
/ homotopy and Cat isomorphism

An element is nontrivial if  $W$  is not  $M$ , or  $f$  is not homotopic to a homeomorphism.

1. Kervaire and Milnor studied this when  $\text{Cat} = \text{Diff}$ , and  $M =$  the sphere: they studied differential structure on the sphere.

They discovered that the answer involved both stable homotopy theory - the hard part of  $\pi_n^s$  (Cokernel  $J$ ) and number theory (Bernoulli numbers)

2. Browder and Novikov discovered that when  $M$  is simply connected and dimension  $> 4$ , this is nontrivial iff there is room for there to be examples explainable via Pontrjagin classes - i.e. iff  $\oplus H^{4i}(M; \mathbf{Q})$  is nonzero for  $4i < \dim(M)$

$4i$  is excluded in this theorem because the Hirzebruch signature theorem tells us that  $p_i$  can be determined from the lower  $p_j$ 's and the homotopy type.

**Theorem:** (Wall, in formulation of Sullivan) There is a semi-long exact sequence (if  $\dim > 4$ ):

$$\dots \rightarrow L_{n+1}(\pi) \rightarrow S^{\text{cat}}(M) \rightarrow [M: F/\text{Cat}] \rightarrow L_n(\pi)$$

where  $F/\text{Cat}$  is related to bundle theory, and  $\pi$  is the fundamental group, and  $L_n$  is made up out of Hermitian forms over  $\mathbf{Z}\pi$  (and their automorphisms).

Actually we should write  $L_n(\pi, w)$  where  $w$  is the orientation character.

**Note that  $S$  is a set,  $[M; F/\text{Cat}]$  is contravariantly functorial, and  $L_n(\pi)$  is covariantly functorial.** Pretty bad!

(Exact sequence is in the sense of pointed sets -  $S(M)$  has the distinguished element of the identity.  $L_{n+1}$  acts on  $S$ .)

There are variants for manifolds with  $\partial$  -- and the notation here is very important in suggesting how to be functorial.

Definition: Let  $M$  be a manifold with boundary, then we define:

$$S^{\text{cat}}(M, \partial M) =$$

$\{(W, f) \mid W \text{ is a Cat-manifold with boundary,}$   
 $f: (W, \partial W) \rightarrow (M, \partial M) \text{ is a simple homotopy equivalence of pairs}\}$

/ homotopy and Cat isomorphism.

$$S^{\text{cat}}(M) = \{(W, f) \mid W \text{ is a Cat-manifold with boundary,}$$

$f: (W, \partial W) \rightarrow (M, \partial M) \text{ is a simple homotopy equivalence of pairs, and } f|_{\partial W}$   
 $\text{is a homeomorphism}\} / \text{homotopy (rel } \partial) \text{ and Cat isomorphism.}$

**Note that the relative theory is denoted by an absolute symbol and the theory without boundary conditions is denoted by the “relative” notation, but really think of it as the theory of a pair.**

This notation gains for us functoriality.

Note that  $[M/\partial M : F/\text{Cat}]$  is functorial w.r.t.  $M$  in the category of manifolds with boundary with codimension zero inclusions as morphisms.

**Theorem:** The surgery exact sequence is functorial for manifolds with codimension 0 inclusions.

There is also an obvious exact sequence (that is semi-infinite) of a pair. This is true for Diff, PL, Top.

Things work better in (PL and) Top. We restrict attention to Top.

**Theorem (Quinn, Ranicki):**  $S^{\text{Top}}(M)$  is covariantly functorial w.r.t. maps  $M \rightarrow N$  that are orientation true and such that  $\dim N - \dim M \geq 0$  and  $\equiv 0 \pmod{4}$ .

Where does this come from?

The hint comes from a **calculation** of Sullivan -- who did remarkable work on the structure of F/O (which shows that this is as hard as the hard part of the stable homotopy groups of spheres) and F/Top.

$$F/\text{Top} \cong K(\mathbb{Z}_2, 2) \times K(\mathbb{Z}_2, 4) \times K(\mathbb{Z}_2, 6) \times K(\mathbb{Z}_2, 8) \times K(\mathbb{Z}_2, 10) \times K(\mathbb{Z}_2, 12) \times \dots \text{at } 2$$

$$F/\text{Top} \cong \text{BO} [1/2]$$

The first line has a clear 4-fold periodicity; BO has an 8-fold periodicity, but away from 2, it has 4-fold.

$$\text{So } \mathbf{Z} \times F/\text{Top} \cong \Omega^4(\mathbf{Z} \times F/\text{Top})$$

This is suggestive of the following result (when combining with the 4-fold periodicity of L-groups):

**Theorem (Siebenmann)** There is an exact sequence

$$0 \rightarrow S^{\text{Top}}(M) \rightarrow S^{\text{Top}}(M \times D^4) \rightarrow \mathbf{Z}.$$

Cappell and I have given a geometric interpretation of this in terms of “branched  $S^1$  fibrations” and the theory of embeddings. This sequence embeds  $S^{\text{Top}}(M)$  in an abelian group, and it is actually the kernel of a homomorphism.

Under the “isomorphism”, L-classes shift by 4-dimensions.

**Theorem (Bryant Ferry Mio W)** There are isomorphisms:

$$S^H(M) \cong S^H(M \times D^4) \cong S^{\text{Top}}(M \times D^4)$$

Here  $S^H(M)$  is a structure set defined using homology manifolds up to s-cobordism.

Not only are there periodicities associated to the trivial bundle, but there are Thom isomorphisms associated to nontrivial (orientable) bundles.

As Grothendieck pointed out in his Riemann-Roch theorem, and Atiyah-Singer exploited in their index theorem, this implies

**Theorem** (BFMW)  $S^H(M)$  is covariantly functorial w.r.t. maps  $M \rightarrow N$  that are orientation true and such that  $\dim N - \dim M \equiv 0 \pmod{4}$ .

This has a lot of implications, of which the following are a few.

- In the periodicity, the push forward of, say, a nontrivial elements of  $S^{\text{Top}}(S^4 \times S^5) \cong \mathbf{Z}$  to  $S^H(S^5)$  shows that manifolds can push forward to nonmanifolds.

These homology manifold homotopy spheres are not *resolvable*. That is there is no map from a manifold to them with Čech contractible inverse images. We conjecture, that against the Bing-Borsuk conjecture, all of these have representatives that are topologically homogeneous (and necessarily nonmanifolds!).

This would be an analogue of a theorem of Edwards.

- $S^H(M)$  can be extended to a homotopy functor of CW complexes. It is a locally linear functor in the sense of Goodwillie, but it is not linear.
- If  $M \rightarrow K(\pi, 1)$  has kernel in some  $H_{4i}(\ ; \mathbf{Q})$ , then  $S^{\text{Top}}(M)$  is infinite, with infinite variation in  $p_i$ .
- For  $\pi$  with torsion and  $M$  of dimension  $3 \pmod{4}$ ,  $S^{\text{Top}}(M)$  is infinite, but not detected by characteristic classes. (**Chang-Weinberger** building on work of Browder-Livesay and Hirzebruch for  $\pi = \mathbf{Z}_2$ , Wall for  $\pi$  finite, and ideas of Cheeger and Gromov related to  $L^2$  cohomology in general).
- Although every orientation preserving PL homeomorphism of a sphere  $S^n$  can be isotoped to the identity -- the growth of the Lipschitz constant of such an isotopy can not be computed in terms of curvature, diameter, and injectivity radius of the sphere (if given a general Riemannian metric,  $n > 4$ ).

The construction of the periodicity map by Cappell-W, depends on a different functoriality for surgery as applied to stratified spaces.

Cappell-W-Yan have a long standing program to try to extend the periodicity and functoriality to the equivariant setting. This is still some way off -- as the methods seem

to be entirely geometric, with little algebraicization feasible. We do have, however, the following general periodicity theorem:

**Theorem** (Weinberger-Yan) If  $V$  is a  $2 \times$  **C-representation** of a compact group, then in the topological setting (allowing homology manifolds, but requiring a mild gap hypothesis on dimensions of fixed sets, and a high dimensionality), we have an isomorphism

$$S^G(M) \cong S^G(M \times V).$$

Which should be a first step towards a general periodicity.

We have other special cases and some applications to geometric topology:

**Theorem** (Cappell-W-Yan). If  $M$  is a  $G$ -manifold, with fixed set  $F$ . If the normal representation is  $2 \times$  **C-representation**, then any manifold homotopy equivalent to  $F$  is the fixed set of a  $G$ -action on manifold  $M'$  homotopy equivalent to  $M$ .

(This is not true in general without the condition on the normal representation.)

Note, it is not a priori obvious even that  $F'$  embeds in  $M'$ ! That is a theorem of Browder-Casson-Haefliger-Sullivan-Wall, which underlies the construction of periodicity.