Computing equivariant stable homotopy classes of maps

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Collaboration

Declaration

Various results of the speaker in this talk were obtained in a collaboration with various subsets of

{Čadek, Filakovský, Krčál, Matoušek, Sergeraert, Wagner}.

Embeddings of simplicial complexes

Consider the following algorithmic problem:

embeddability problem

Given a finite simplicial complex K of dimension k, does there exist an embedding of K into $\mathbb{R}^n$?

The answer depends on the relation between k and n:

Theorem (Matoušek, Tancer and Wagner)

- $n = 2k + 1$: *always exists*
- $n = 2k$: *decidable in polynomial time (computation of the "van Kampen obstruction")*
- $n < \frac{3}{2}(k + 1)$: *NP-hard (reduction to 3-SAT)*
- $n = k, k + 1$: *undecidable (reduction to sphere recognition)*

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Weber theorem

Theorem (Weber)

When $n \geq \frac{3}{2}(k+1)$, the embeddability is equivalent to the existence of a $\mathbb{Z}/2$-equivariant map

$$(K \times K) \setminus \Delta_K \to S^{n-1}.$$

More generally, we ask the following question:

existence of an equivariant map

Given two spaces X, Y with free actions of a finite group G, does there exist an equivariant map $X \to Y$?

We attack this existence problem via homotopy theory.

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More generally, we ask the following question:

compute equivariant homotopy classes

Given two spaces X, Y with free actions of a finite group G, compute the set of equivariant homotopy classes $[X, Y]_G$.

We attack this existence problem via homotopy theory.

Stability assumption

Theorem (Čadek, Krčál, Matoušek, Vokřínek, Wagner)

The problem of the existence of a continuous extension

$$\begin{array}{ccc} A & \longrightarrow & Y \\ \big\downarrow & \nearrow & \\ X & & \end{array}$$

is undecidable for each $d = \operatorname{conn} Y$.

In the proof, we use extension problems with $\dim X = 2d + 2$. We do not expect solvability of the existence problem in this situation. (It is enough to consider $Y = S^{d+1}$ for d is odd; the case $Y = S^{d+1}$ with d even seems solvable!)

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compute equivariant homotopy classes (stable version)

Given two spaces X, Y with free actions of a finite group G and with $\dim X \leq 2 \operatorname{conn} Y$, compute $[X, Y]_G$.

We obtained the following result

Theorem (Čadek, Krčál, Vokřínek)

The above problem is solvable in polynomial time (G and d fixed). In particular, the embeddability problem with $n \geq \frac{3}{2}(k+1)$ is decidable in polynomial time.

It is also possible to test homotopy of maps, even non-stably but with Y simply connected [Filakovský, Vokřínek].

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Group structure

The meaning of "compute $[X, Y]_G$" comes from the following fact:

important fact

Stably, $[X, Y]_G$ is equipped with a structure of an abelian group without a definite choice of the zero. (It is an "abelian heap".)

The abelian group $[X, Y]_G$ is computed by induction over the "Postnikov tower" of Y.
Since Y has a *free* action of G, we really work with a Moore–Postnikov tower of Y over $B = EG$.
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Lifting-extension problem

lifting-extension problem (stable version)

Given a "stable" commutative square

$$\begin{array}{ccc} A & \longrightarrow & Y \\ {\scriptstyle\iota}\big\downarrow & & \big\downarrow{\scriptstyle\varphi} \\ X & \longrightarrow & B \end{array}$$

compute the set $[X, Y]^A_B$ of equivariant homotopy classes of diagonals.

(Stability means $\dim X \leq 2d$, where $d = \operatorname{conn}\operatorname{fib}\varphi$.)

In the border case $\dim X = 2d + 1$, the existence is to be decided.

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Theorem (Čadek, Krčál, Vokřínek)

The stable lifting-extension problem can be solved in polynomial time (G and d fixed).

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Outline of the algorithm

Again, $[X, Y]^A_B$ is an abelian group. We approximate Y by its Moore–Postnikov tower over B with stages P_n.

Then $[X, P_n]^A_B$ is related to $[X, P_{n-1}]^A_B$ via an "exact sequence"

$$[\Delta^1 \times X, P_{n-1}]^{A'}_B \to [X, B \times K(\pi_n, n)]^A_B \to [X, P_n]^A_B \to$$
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from which it is possible to compute $[X, P_n]^A_B$ inductively ($[X, B \times K(\pi_n, n)]^A_B \cong H^n_G(X, A; \pi_n)$).

main obstacles

- Construct the Moore–Postnikov tower.
- Construct the abelian group structure on $[X, P_n]^A_B$.

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Simplicial sets with effective homology

In general, the Moore–Postnikov stages P_n are *infinite* simplicial sets. To construct the tower, we need their homotopy groups, computed through homology.

simplicial sets with effective homology (Sergeraert)

Thus, we construct P_n together with a chain homotopy equivalence $C_* P_n \Longleftrightarrow C_*^{\mathrm{ef}} P_n$, where $C_*^{\mathrm{ef}} P_n$ is some locally finite chain complex, where we perform all homological computations.

effective homology of Eilenberg–MacLane spaces

This chain homotopy equivalence is constructed inductively, starting from $K(\pi_n, n)$. Following the work of Eilenberg and MacLane, we use $K(\pi_n, n) = \overline{W} K(\pi_n, n-1)$ for induction (using bar construction) and the simple case $K(\pi_n, 1)$. Their version is not polynomial time, however – solved by Krčál, Matoušek, Sergeraert.

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Equivariant chain homotopy equivalences

To make things equivariant requires a new idea, since the effective homology of $K(\pi_n, 1)$ is not equivariant.

Theorem (Vokřínek)

Given a chain homotopy equivalence $M \Longleftrightarrow N$ and a free action of G on M, it is possible to construct an equivariant chain homotopy equivalence $M \Longleftrightarrow N'$.

Idea of a proof.

Equip N with an action of G up to a coherent system of homotopies. Construct a "strictification" N' that has a (strict) action of G. Then $M \Longleftrightarrow M' \Longleftrightarrow N'$. Work required to make this algorithmic and *finite* (the system of homotopies is infinite). □

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Addition in $[X, P_n]^A_B$

We assume that P_n has a "zero section" and that the image of A lies on this zero section.

The addition in $[X, P_n]^A_B$ comes from a *weak* fibrewise H-space structure on P_n:

$$\begin{array}{ccc} P_n \vee_B P_n & \longrightarrow & P_n \\ \downarrow & \nearrow & \\ P_n \times_B P_n & & \end{array}$$

We are not able to make $P_n \vee_B P_n$ into a simplicial set with effective homology. This leads us to a relaxed version of an H-space, with zero only up to a homotopy.

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Weak H-space structure

The map $P_n \mathbin{\widetilde{\times}_B} P_n \to P_n$ is made of four pieces:

- addition $P_n \times_B P_n \to P_n$, denoted $x + y$,
- left zero homotopy $\lambda \colon \Delta^1 \times (B \times_B P_n) \to P_n$, $\lambda_x \colon x \sim 0 + x$,
- right zero homotopy $\rho \colon \Delta^1 \times (P_n \times_B B) \to P_n$, $\rho_x \colon x \sim x + 0$,
- second order homotopy $\Delta^2 \times (B \times_B B) \to P_n$, balancing the left and right zero homotopies.

$$P_n \mathbin{\widetilde{\times}_B} P_n = (\Delta^2 \times X) \cup (d_1\Delta^2 \times X_0) \cup (d_0\Delta^2 \times X_1) \cup (2 \times X_2).$$

Weak H-space structure

The map $P_n \mathbin{\widetilde{\times}_B} P_n \to P_n$ is made of four pieces:

- addition $P_n \times_B P_n \to P_n$, denoted $x + y$,
- left zero homotopy $\lambda\colon \Delta^1 \times (B \times_B P_n) \to P_n$, $\lambda_x\colon x \sim 0 + x$,
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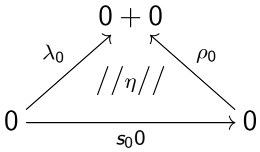

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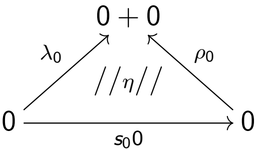

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Addition in $[X, P_n]^A_B$, part 2

This weakening makes nicely for an algorithmic construction but does not allow simple minded addition of homotopy classes.

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If $\ell_0, \ell_1 \colon X \to P_n$ are zero on A then $\ell_0 + \ell_1$ takes values $0 + 0 \neq 0$ on A. There is however a homotopy (two in fact) $0 \sim 0 + 0$ and by extending it to the whole of X we obtain a homotopy $\ell \sim \ell_0 + \ell_1$ and ℓ represents an element $[\ell] = [\ell_0] + [\ell_1] \in [X, P_n]^A_B$. This is made algorithmic rather easily.

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Summary

The important ingredients are:

- Simplicial sets with effective homology and constructions with them.
- The effective homology of Eilenberg–MacLane spaces – polynomial-time and equivariant.
- The weak H-space structure on the stable part of the Moore–Postnikov tower.
- The induced addition on homotopy classes and computations with exact sequences.
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