On the Product in Negative Tate Cohomology

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Introduction

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The grading of this product is given by:

$$H_k(BG,\mathbb{Z}) \otimes H_l(BG,\mathbb{Z}) \to H_{k+l+dim(G)+1}(BG,\mathbb{Z})$$

(where $k, l > 0$ in case G is discrete).

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- The construction of the product due to Kreck.
- The relation of the product, in the case of finite groups, to the product in negative Tate cohomology.
- Computational results for compact Lie groups.

The geometric view point - the Kreck product

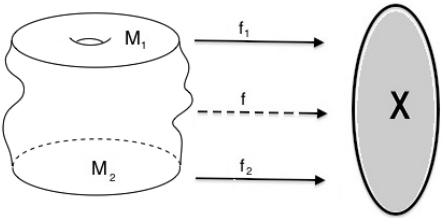
The product is given in the language of stratifolds and stratifold homology. This is a geometric way to describe integral homology as a bordism theory.

Oriented bordism

We start with a reminder of the definition of oriented bordism. The oriented bordism groups of a space X are given by

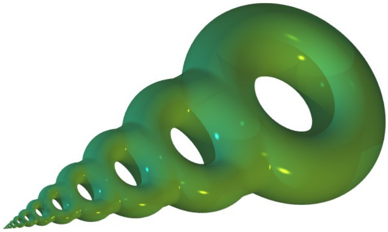
$$\Omega_k^{SO}(X) = \{(f : M \to X)\} / \sim$$

where M is a closed oriented k-manifold, $f : M \to X$ is a continuous map and the relation is bordism.



Integral homology and cohomology via stratifolds

To obtain integral homology one replaces manifolds with stratifolds. Stratifolds were defined by Kreck. These are generalization of manifolds: topological spaces together with a sheaf of real functions.



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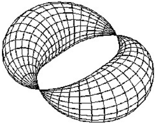

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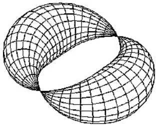

A stratifold is called oriented if its top stratum is oriented and its codimension one stratum is empty.

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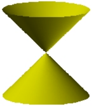
- Real algebraic varieties



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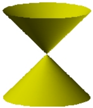

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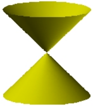

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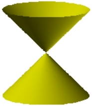

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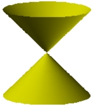

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- The one point compactification of the interior of a smooth manifold with boundary.
- The product of stratifolds
- The cone over a stratifold

Stratifold homology

The stratifold homology groups of X are:

$$SH_k(X) = \{(f : S \to X)\} / \sim$$

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Theorem [Kreck]

There is a natural isomorphism $SH_k(X) \cong H_k(X, \mathbb{Z})$

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Notice, that $\tilde{S}$ is a closed oriented stratifold of dimension $dim(S) + dim(G)$ together with a free and orientation preserving G action.

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On the other hand, given a compact oriented stratifold $\tilde{S}$ with an orientation preserving free G action, then its quotient $S = \tilde{S}/G$ comes with a classifying map to BG, which is the unique map (up to homotopy) s.t. the following square commutes:

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This can also be viewed as odd spheres with the antipodal $\mathbb{Z}/2$ action.

The product in the homology of BG

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The first construction will be the product

$$H_k(BG,\mathbb{Z}) \otimes H_l(BG,\mathbb{Z}) \to H_{k+l+dim(G)}(BG,\mathbb{Z})$$

given by:

$$[S] \times [S'] = [S \times S']$$

On the right side the action of G is the diagonal action.

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This defines a product:

$$H_k(BG,\mathbb{Z}) \otimes H_l(BG,\mathbb{Z}) \to H_{k+l+dim(G)+1}(BG,\mathbb{Z})$$

The product in the homology of BG

Here is an example, where this product does not vanish:
Let $G = \mathbb{Z}/2$, and take the generators for $H_k(BG, \mathbb{Z})$ and $H_l(BG, \mathbb{Z})$.

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Restricting to integral coefficients we get:

$$H_k(BG, \mathbb{Z}) \otimes H_l(BG, \mathbb{Z}) \to H_{k+l+1}(BG, \mathbb{Z})$$

The product in the homology of BG

The following theorem relates the Kreck product with the product in negative Tate cohomology.

Theorem

Let G be a finite group, then there is a natural isomorphism between SH_n^G and $\hat{H}^{-n-1}(G,\mathbb{Z})$ and this isomorphism respects the product.

The product in Tate cohomology

In general, this product is non trivial. One family of groups where the product does not vanish is the family of groups with periodic cohomology.Those are exactly the groups such that every Abelian subgroup is cyclic. This includes, for example, all groups having a free and orientation preserving action on some sphere, but not only.

The product in Tate cohomology

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This is the case, for example, if there is a p-Sylow subgroup with non cyclic center.

Computations for Lie groups

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Proposition

Let $G = G_1 \times G_2$. Suppose $\alpha = \alpha_1 \times \alpha_2$ and $\beta = \beta_1 \times \beta_2$, then $\alpha * \beta = 0$ if at least 3 of the following are positive:
$|\alpha_1| + dim(G_1), |\alpha_2| + dim(G_1), |\beta_1| + dim(G_2), |\beta_1| + dim(G_2)$.

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Which implies the following:

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The product vanishes for the $U(n)$, $SU(n)$ and $Sp(n)$ for $n > 2$.

Computations for Lie groups

Proposition

Let $\phi : G \to G'$ be a surjective homomorophism with finite kernel of order n, then for the induced map:

$$\phi_* : H_*(BG, \mathbb{Z}) \to H_*(BG', \mathbb{Z})$$

we have:

$$\phi_*(\alpha * \beta) = n \cdot \phi_*(\alpha) * \phi_*(\beta)$$

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$SO(3) \cong \mathbb{R}P^3$ - $H_{(*+2)}(BG, \mathbb{Q})$ is the ideal in $\mathbb{Q}[X]$ generated by X with $|X| = 4$.