Cofibration Categories and Quasicategories

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“Classification” of models of homotopy theories

Approaches to abstract homotopy theory fall into two types.

Classical:

- Quillen model categories,
- Thomason model categories,
- (co)fibration categories. . .

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Main goal

The homotopy theory of cofibration categories is equivalent to the homotopy theory of cocomplete quasicategories.

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$$A_0 \overset{(\sim)}{\rightarrowtail} A_1 \overset{(\sim)}{\rightarrowtail} A_2 \overset{(\sim)}{\rightarrowtail} \dots \overset{(\sim)}{\dashrightarrow} A_\infty.$$

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- the category of topological spaces with weak homotopy equivalences and Dold–Serre fibrations.

Overview of quasicategories

A *quasicategory* is a simplicial set $\mathcal{A}$
with fillers for all *inner horns* $(0 < i < m)$.

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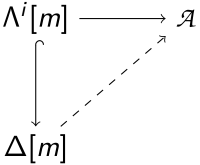

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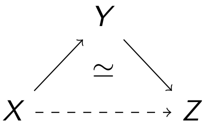
- 1-simplices: morphisms
- higher simplices: homotopies



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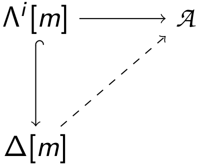

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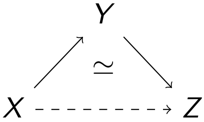

Examples

- Nerves of categories.
- Kan complexes (in particular $\operatorname{Sing} X$ for a space X).

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mapping spaces		
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homotopy colimits	derived colimits "there is a unique x such that. . . "	universal properties "the space of x such that. . . is contractible"

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isomorphisms

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$$\Big\downarrow F$$

$$\mathcal{D} \qquad FA \xrightarrow{\cong} X$$

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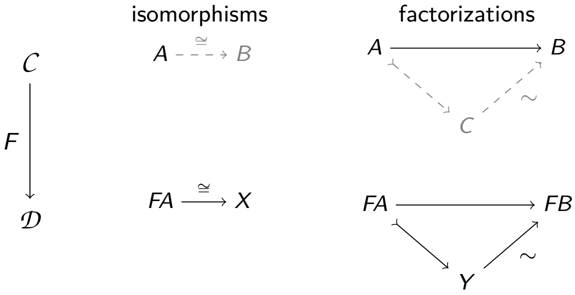

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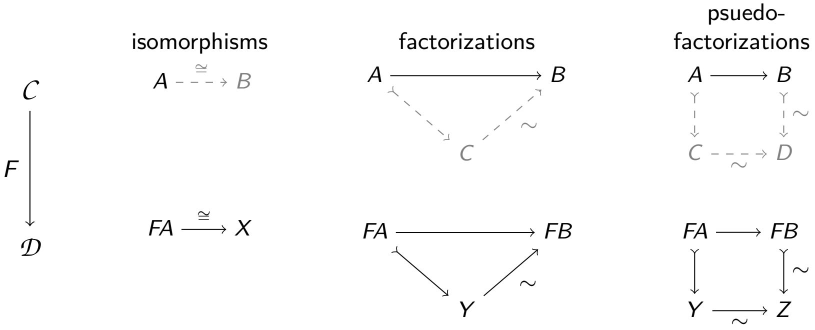

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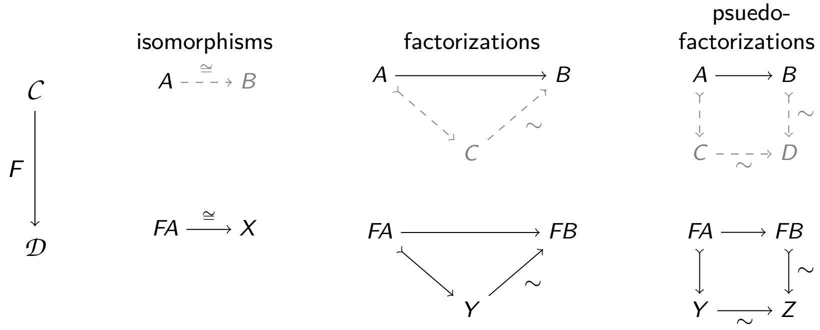

Theorem

With the above weak equivalences and fibrations the category of small cofibration categories is a fibration category.

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The category of cocomplete quasicategories with categorical equivalences and inner isofibrations is a fibration category.

Main result

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The fibration category of cofibration categories
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Work in progress

The case of finitely cocomplete cofibration categories
and quasicategories.

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$$\begin{array}{ccccc} & & X_1 & & \\ & \overset{\sim}{\swarrow} & & \searrow & \\ & X_{01} & & X_{12} & \\ \nearrow & & & & \overset{\sim}{\nwarrow} \\ X_0 & & & & X_2 \end{array}$$

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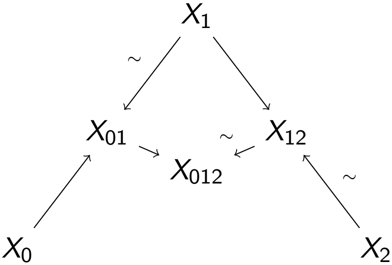

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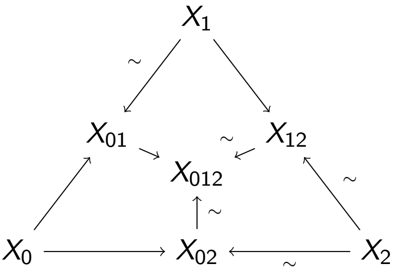

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- $\mathsf{N_f}$ induces an equivalence of homotopy categories.

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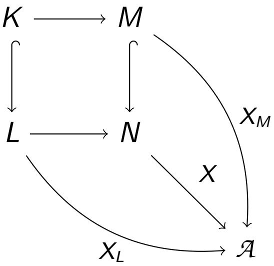

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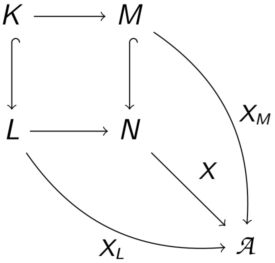

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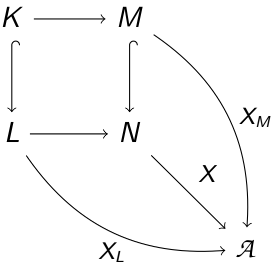

- $N = L \sqcup_K M$
- $\operatorname{colim}_N X \simeq$ $\operatorname{colim}_L X_L \sqcup_{\operatorname{colim}_K X_K} \operatorname{colim}_M X_M$

The category of countable diagrams $D^c\mathcal{A}$

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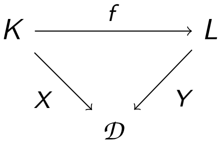

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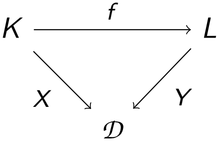

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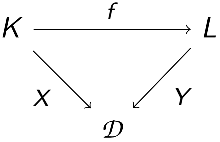

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Theorem

If $\mathcal{A}$ is countably cocomplete, then $D^c\mathcal{A}$ is a cofibration category.

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Theorem

The functor

$$\operatorname{Ho} D^c \colon \operatorname{Ho} \mathrm{QCat}^c \to \operatorname{Ho} \mathrm{CofCat}$$

is an inverse to $\operatorname{Ho} \mathrm{N_f}$.