coMalcev Monoidal Monads

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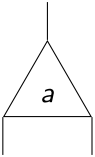
Signatures



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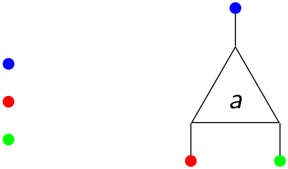

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- Which are typed

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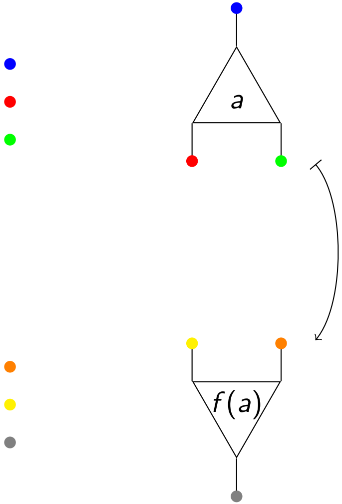

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  1. Map of function symbols

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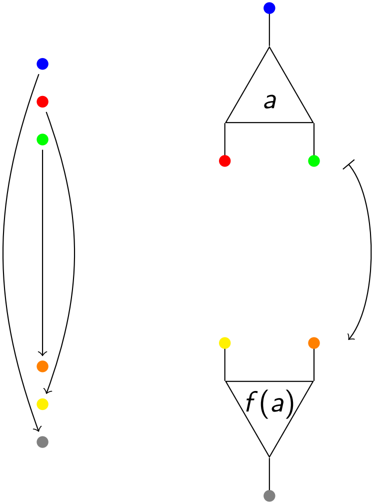

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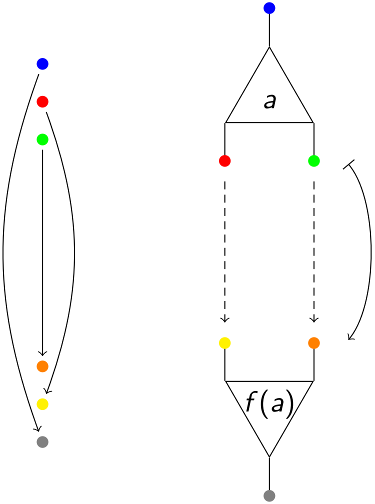

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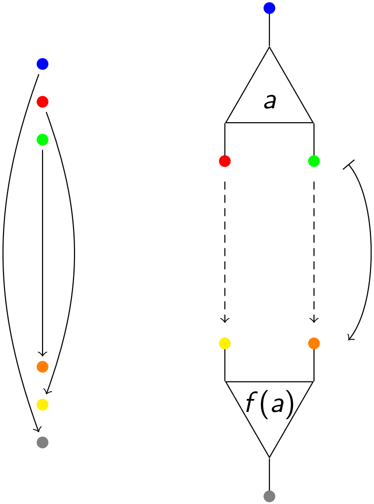
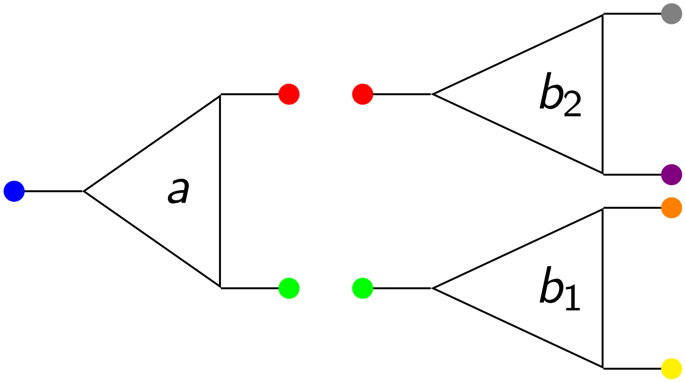


- Signatures consist of function symbols
- Which are typed
- Morphisms:
  1. Map of function symbols
  2. Map of types
  3. Types must match
- The category will be called **Sig**
- **Sig** is fibered over **Set**

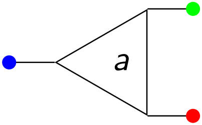
Monoidal Structure on Signatures

$$A \otimes B :$$



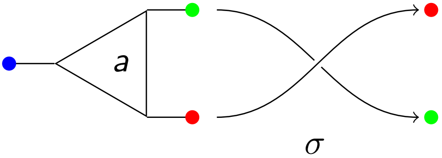
The Symmatrization Monad $\mathcal{S}$

$$a \quad \in \quad A$$



The Symmatrization Monad $\mathcal{S}$

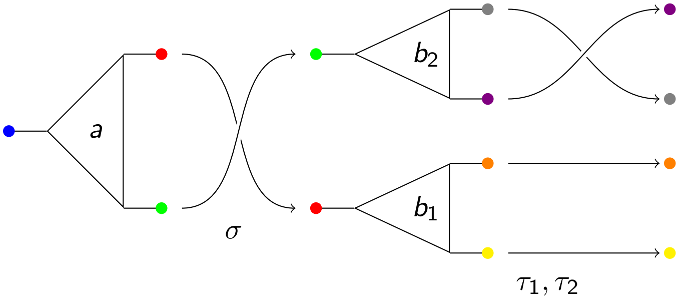
$$(a, \sigma) \in \mathcal{S}A$$



This defines a monad **Sig** $\xrightarrow{\mathcal{S}}$ **Sig**.
Multiplication is composition of permutations.

Monoidal Structure on $\mathcal{S}$

$$\mathcal{S}A \otimes \mathcal{S}B \xrightarrow{\varphi} \mathcal{S}(A \otimes B)$$



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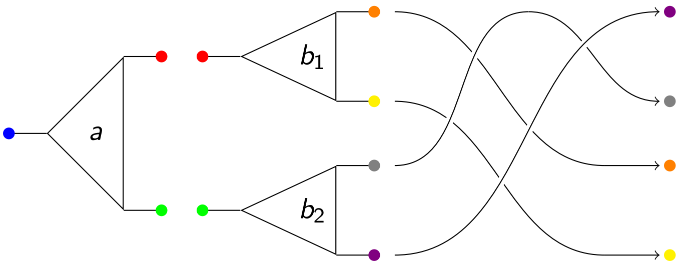

$\sigma * (\tau_1, \tau_2)$

Motivation

- Symmetric signatures, $\mathbf{Sig}^{\mathcal{S}}$, are equivalent to analytic functors

$$\begin{tikzcd}
 & \mathbf{Sig}^{\mathcal{S}} \simeq \mathbf{An} \arrow[dl] \\
\mathbf{Sig} \arrow[loop left, "\mathcal{S}"] \arrow[d] & \\
\mathbf{Set} &
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- Signatures with amalgamation are equivalent to polynomial functors
- Composition of functors gives tensors on both
- Kleisli object is monoidal because $\mathcal{S}$ is lax monoidal
- Understand the emergence of $\ddot{\otimes}$ from $\mathcal{S}$, not functors

$$\begin{tikzcd}
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coMalcev Operations

in monoidal categories

$$\begin{tikzcd}
X \times X \arrow[r, "\delta \times 1_X"] \arrow[dr, "! \times 1_X"'] & X \times X \times X \arrow[d, "\zeta"] & X \times X \arrow[l, "1_X \times \delta"'] \arrow[dl, "1_X \times !"] \\
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- Malcev operation

coMalcev Operations

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$$\begin{tikzcd}
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- Add comonoid structure

coMalcev Operations

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& X \arrow[ul, "\eta \otimes 1_X"] \arrow[u, "\zeta"'] \arrow[ur, "1_X \otimes \eta"'] &
\end{tikzcd}$$

- Malcev operation
- Switch × to $\otimes$
- Add comonoid structure
- Reverse arrows

Examples of coMalcev Monads

monoidal and not monoidal

- $G \times (-) \colon \mathbf{Set} \to \mathbf{Set}$, for a group G: $(g, x) \overset{\zeta}{\mapsto} (g, g^{-1}, g, x)$

- $H \otimes (-) \colon \mathcal{C} \to \mathcal{C}$, for a Hopf algebra H

- The Jonsson-Tarski monad admits a coMalcev operation
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The last example is given by

$$\begin{array}{ccc} \mathcal{S}A & \xrightarrow{\zeta_A} & \mathcal{S}^3 A \\ (a, \sigma) & \mapsto & (a, \sigma, \sigma^{-1}, \sigma) \end{array}$$

General Setup

of algebraic 2-categories

To state the general theorem we need a convenient setting to develop a bit of algebra.

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Definition

An algebraic 2-category is a 2-category $\mathcal{D}$, which admits

- Finite products
- Objects of monoids and actions
- Eilenberg-Moore objects
- Kleisli objects which commute with above constructions

We also require that 0-cells $X \in \mathcal{D}$ have reflexive coequalizers.

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Last condition means that $\mathcal{D}(-, X)$ takes values in categories with reflexive coequalizers and functors preserving them

The Main Theorem

Theorem

Let $\mathcal{D}$ be an algebraic 2-category. Then:

1. $p_{\mathcal{D}} \colon Act(\mathcal{D}) \to Mon(\mathcal{D})$ *is a 2-fibration (in the sense of Hermida)*
2. *Both* $Act(\mathcal{D})$ *and* $Mon(\mathcal{D})$ *admit Eilenberg-Moore objects of coMalcev monads*
3. $p_{\mathcal{D}}$ *preserves them*

Elements of the Proof for $Mon(\mathcal{D})$

Constructing the Tensor

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Constructing the Tensor

The EM-objects already exist – we only add a monoidal structure.
We define $\ddot{\otimes}$ by Linton's formula in $\mathcal{D}(-, X)$:

$$\begin{tikzcd}
\mathcal{S}(\mathcal{S}A \otimes \mathcal{S}B) \arrow[rr, "\mathcal{S}(a \otimes b)"] \arrow[dr, "\mathcal{S}(\varphi)"'] & & \mathcal{S}(A \otimes B) \arrow[r, "\mathrm{coeq}"] & A \ddot{\otimes} B \\
& \mathcal{S}^2(A \otimes B) \arrow[ur, "\mu"'] & & \\
& \mathcal{S}^3(A \otimes B) \arrow[dr, "\mathcal{S}(\mu)"] & & \\
\mathcal{S}^2(\mathcal{S}A \otimes \mathcal{S}B) \arrow[uuu, "\mu"] \arrow[ur, "\mathcal{S}^2(\varphi)"] \arrow[rr, "\mathcal{S}^2(a \otimes b)"'] & & \mathcal{S}^2(A \otimes B) \arrow[uuu, "\mu"'] \arrow[r, "\mathrm{coeq}"] & \mathcal{S}(A \ddot{\otimes} B) \arrow[uuu, "a \ddot{\otimes} b"']
\end{tikzcd}$$

Elements of the Proof for $Mon(\mathcal{D})$

Associativity

coMalcev allows us to invert the associativity map. It connects the iterated tensor with the “triple Linton formula”:

$$\begin{tikzcd}
\mathcal{S}(\mathcal{S}A \otimes \mathcal{S}(\mathcal{S}^2 B \otimes \mathcal{S}^2 C)) \arrow[r, "\mathcal{S}(1 \otimes \mathcal{S}(\varphi))"] & \mathcal{S}(\mathcal{S}A \otimes \mathcal{S}^2(\mathcal{S}B \otimes \mathcal{S}C)) \arrow[r, bend left, "\mu \circ \mathcal{S}(\varphi) \circ \mathcal{S}(1 \otimes \mathcal{S}(\mu)) \circ \mathcal{S}(1 \otimes \mathcal{S}^2(\varphi))"] \arrow[r, bend right, "\mathcal{S}(a \otimes \mu) \circ \mathcal{S}(1 \otimes \mathcal{S}^2(b \otimes c))"'] \arrow[d, bend left, "\mu \circ \mathcal{S}(\varphi) \circ \mathcal{S}(\eta \otimes \mu)"'] & \mathcal{S}(A \otimes \mathcal{S}(B \otimes C)) \arrow[r] \arrow[d, bend right, "\mu \circ \mathcal{S}(\varphi) \circ \mathcal{S}(\eta \otimes 1)"] & A \ddot{\otimes} (B \ddot{\otimes} C) \arrow[d, bend right] \\
\mathcal{S}(\mathcal{S}A \otimes (\mathcal{S}^3 B \otimes \mathcal{S}^3 C)) \arrow[u, "\mathcal{S}(1 \otimes \varphi)"'] & \mathcal{S}(\mathcal{S}A \otimes (\mathcal{S}B \otimes \mathcal{S}C)) \arrow[l, red, "\mathcal{S}(1 \otimes (\zeta \otimes \zeta))"'] \arrow[u, red, dashed, bend left] \arrow[r, bend left, "\mu \circ \mathcal{S}(\varphi) \circ \mathcal{S}(1 \otimes \varphi)"] \arrow[r, bend right, "\mathcal{S}(a \otimes b \otimes c)"'] & \mathcal{S}(A \otimes (B \otimes C)) \arrow[r] \arrow[u, bend right, "\mathcal{S}(1 \otimes \eta)"'] & A \widehat{\otimes} (B \widehat{\otimes} C) \arrow[u, red, dashed, bend right]
\end{tikzcd}$$

Applications

Set $\mathcal{D} = \mathbf{Fib}/\mathbf{Set}$

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Act(\mathbf{Fib}/\mathbf{Set}) =: \mathbf{Almf}/\mathbf{Set} \arrow[d, "p"] \\
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$$\begin{tikzcd}
Act(\mathbf{Fib}/\mathbf{Set}) =: \mathbf{Almf}/\mathbf{Set} \arrow[dd, "p"] & \ni & \mathbf{Sig} \times \mathbf{Set}^{\to} \arrow[loop right, "\mathcal{S} \times \underline{G}"] \arrow[d, "*"] \\
 & & \mathbf{Set}^{\to} \arrow[loop below, "\underline{G}"] \\
Mon(\mathbf{Fib}/\mathbf{Set}) =: \mathbf{Lmf}/\mathbf{Set} & \ni & \mathbf{Sig} \arrow[loop right, "\mathcal{S}"] \arrow[d] \\
 & & \mathbf{Set}
\end{tikzcd}$$

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$$\begin{tikzcd}
& \mathbf{Sig}\times\mathbf{Set}^{\to} \arrow[d, "*"] \arrow[loop right, "\mathcal{S}\times\underline{G}"] & & \mathbf{Sig}^{\mathcal{S}}\times G-\mathbf{Set}^{\to} \arrow[d, "*"] \\
Act(\mathbf{Fib}/\mathbf{Set}) =: \mathbf{Almf}/\mathbf{Set} \ni \arrow[dd, "p"] & \mathbf{Set}^{\to} \arrow[loop below, "\underline{G}"] & {} \arrow[r, mapsto] & G-\mathbf{Set}^{\to} \\
& \mathbf{Sig} \arrow[d] \arrow[loop right, "\mathcal{S}"] & & \mathbf{Sig}^{\mathcal{S}} \arrow[d] & \ddot{\otimes} \\
Mon(\mathbf{Fib}/\mathbf{Set}) =: \mathbf{Lmf}/\mathbf{Set} \ni & \mathbf{Set} & {} \arrow[r, mapsto] & \mathbf{Set}
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 & & \mathbf{Sig}\times\mathbf{Set}^{\to} \arrow[d, "*"] \arrow[loop right, "\mathcal{S}\times\underline{G}"] & & \mathbf{Sig}^{\mathcal{S}}\times G-\mathbf{Set}^{\to} \arrow[d, "*"] \\
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 & & \mathbf{Set}^{\to} \arrow[loop below, "\underline{G}"] & & G-\mathbf{Set}^{\to} \\
 & & {\color{red}Gph(T)} \arrow[d] \arrow[loop right, "\mathcal{S}"] & & {\color{red}Gph(T)^{\mathcal{S}}}\ \ddot{\otimes} \arrow[d] \\
Mon(\mathbf{Fib}/\mathbf{Set}) =: \mathbf{Lmf}/\mathbf{Set} & \ni & {} \arrow[rr, mapsto] & & {} \\
 & & \mathbf{Set} & & \mathbf{Set}
\end{tikzcd}$$

T strongly regular