Eilenberg-Moore categories and Kan-injectivity

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Samuel Eilenberg Centenary Conference, Warsaw, July 22-26, 2013

$\mathcal{X}$ poset enriched category:

Hom(X, Y) are posets, and, for $X \xrightarrow{f} Y \underset{h}{\overset{g}{\rightrightarrows}} Z \xrightarrow{k} W$, $g \leq h \Rightarrow kgf \leq khf$

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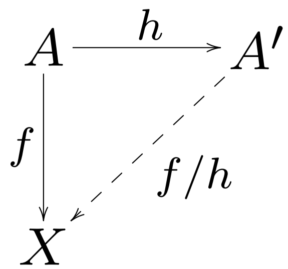
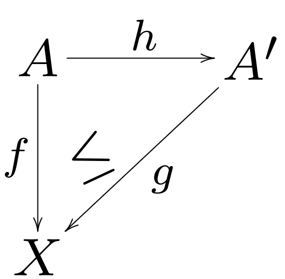
X is left-Kan injective w.r.t. $h : A \to A'$ if every $f : A \to X$ admits f/h:



(1) $f = (f/h) \cdot h$

(2) If



then $f/h \leq g$.

Examples of injectivity in $\mathbf{Top}_0$:

$\mathcal{H} \subseteq$ continuous maps	spaces injective wrt $\mathcal{H}$
embeddings	continuous lattices [D. Scott, 1972]

Examples of injectivity in $\mathbf{Loc}$:

$\mathcal{H} \subseteq$ localic maps	spaces injective wrt $\mathcal{H}$
one-to-one which preserve finite suprema	stably locally compact locales (=retracts of coherent locales) [P. Johnstone, 1981]

Examples of injectivity in $\mathbf{Top}_0$:

$\mathcal{H} \subseteq$ continuous maps	spaces injective wrt $\mathcal{H}$
embeddings	continuous lattices [D. Scott, 1972]
dense embeddings	continuous Scott domains [D. Scott, 1980]

Examples of injectivity in $\mathbf{Loc}$:

$\mathcal{H} \subseteq$ localic maps	spaces injective wrt $\mathcal{H}$
one-to-one	stably supercontinuous lattices [B. Banaschewski, 1985]
one-to-one which preserve finite suprema	stably locally compact locales (=retracts of coherent locales) [P. Johnstone, 1981]

M. Escardó and others in a number of papers in the late 90's observed that:

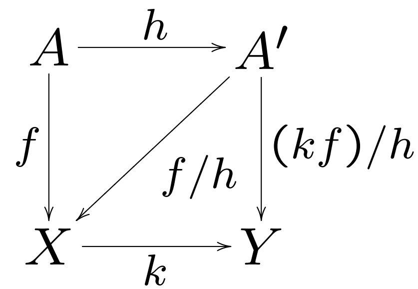
In these examples, and others, the Kan-injective spaces are just the Eilenberg-Moore algebras of a Kock-Zöberlein (KZ) monad [A. Kock, 1995].

[M.Carvalho and L.S., 2011]:

Kan-injectivity also for morphisms

$X \xrightarrow{k} Y$ is left Kan-injective w.r.t. $A \xrightarrow{h} A'$ if X and Y are so, and, for every $A \xrightarrow{f} X$, we have

$$(kf)/h = k(f/h)$$



Given $\mathcal{H} \subseteq \mathrm{Mor}(\mathcal{X})$,

LInj $\mathcal{H}$:= subcategory of all objects and morphisms left Kan-injective w.r.t. all morphisms of $\mathcal{H}$

Left Kan-injective subcategories (i.e., of the form LInj $\mathcal{H}$) are <u>non-full</u>, in general.

A subcategory $\mathcal{S}$ of $\mathcal{X}$ is said to be closed under left adjoint retracts if, for every commutative square, with $g \in \mathcal{S}$,

$$\begin{array}{ccc} X & \xrightarrow{g \in \mathcal{S}} & Y \\ {\scriptstyle e}\downarrow & & \downarrow{\scriptstyle e'} \\ Z & \xrightarrow[g']{} & W \end{array}$$

with e and e' left adjoint retractions ($ed = 1_Z$ and $1_Y \leq de$)

the morphism g' belongs to $\mathcal{S}$.

> Subcategories $\mathsf{LInj}\,\mathcal{H}$ are closed under left adjoint retracts.

A subcategory $\mathcal{A}$ of $\mathcal{X}$ is said to be KZ-reflective if it is reflective and the left adjoint $F : \mathcal{A} \to \mathcal{X}$ is locally monotone and fulfils the inequality

$$F\eta_X \leq \eta_{FX}, \qquad \text{for every } X \in \mathcal{X}.$$

> A subcategory $\mathcal{A}$ of $\mathcal{X}$ is an Eilenberg-Moore category for a KZ-monad over $\mathcal{X}$ iff it is KZ-reflective and closed under left adjoint retracts.

These subcategories are always of the form $\mathrm{LInj}\,\mathcal{H}$.

Conversely: When is $\mathrm{LInj}\,\mathcal{H}$ an Eilenberg-Moore category for a KZ-monad?

(Left) Kan-injective subcategory problem:

When is $\mathsf{LInj}\,\mathcal{H}$ a KZ-reflective subcategory?

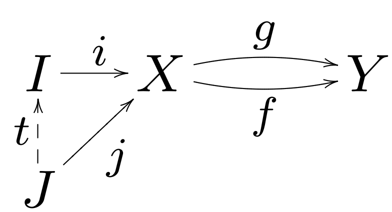
Joint work with Jiří Adámek and Jiří Velebil

> Left Kan-injective subcategories are closed under weighted limits.
>
> In particular, they are closed under inserters.

Given a pair of morphisms $X \underset{f}{\overset{g}{\rightrightarrows}} Y$ in $\mathcal{X}$, the inserter of f and g, denoted ins(f, g), is a morphism $i : I \to X$ such that

(1) $f \cdot i \leq g \cdot i$

(2) If $j : J \to X$ also fulfils $f \cdot j \leq g \cdot j$ then there is a unique $t : J \to I$ such that $j = it$.



(3) i is an order-monomorphism, that is, $i \cdot a \leq i \cdot b \;\Rightarrow\; a \leq b$.

A subcategory $\mathcal{A}$ of $\mathcal{X}$ is said to be an inserter-ideal if for every inserter $i = \mathrm{ins}(f, g)$ in $\mathcal{X}$

$$I \xrightarrow{i} X \underset{f}{\overset{g}{\rightrightarrows}} Y$$

if f belongs to $\mathcal{A}$, then also $i : I \to X$ belongs to $\mathcal{A}$.

Left Kan-injective subcategories are inserter-ideals.

Every reflective, inserter-ideal subcategory is KZ-reflective.

Consequently:

LInj $\mathcal{H}$ reflective $\Leftrightarrow$ LInj $\mathcal{H}$ KZ-reflective $\Leftrightarrow$ LInj $\mathcal{H}$ an Eilenberg-Moore category for a KZ-monad

Left Kan-injective subcategory problem:

When is $\mathsf{LInj}\,\mathcal{H}$ a reflective subcategory?

Kan-Injective Reflection Construction (for a set $\mathcal{H} \subseteq \mathsf{Mor}(\mathcal{X})$)

Goal: To obtain a reflection of X into $\mathsf{LInj}\,\mathcal{H}$

$$X = X_0 \xrightarrow{x_{01}} X_1 \xrightarrow{x_{12}} X_2 \longrightarrow \cdots$$

assuming that $\mathcal{X}$ has weighted colimits:

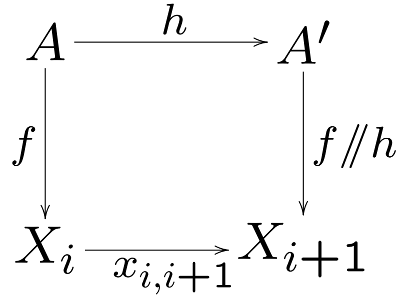
$X_0 = X$.

For i a limit ordinal, $X_i = \mathsf{Colim}_{j<i} X_j$.

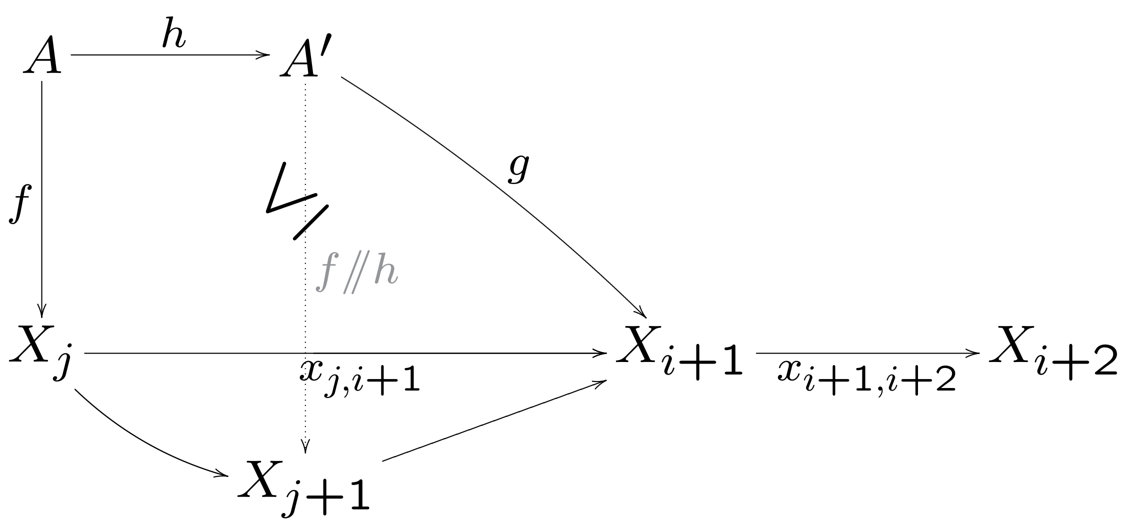
For i even, steps $i \mapsto i+1$ and $i+1 \mapsto i+2$ as follows:

Kan-Injective Reflection Construction. $X = X_0 \xrightarrow{x_{01}} X_1 \xrightarrow{x_{12}} X_2 \longrightarrow \cdots$

Step $i \mapsto i+1$. $x_{i,i+1}$ is the wide pushout of all pushouts of $h \in \mathcal{H}$ along some f with codomain X_i:



Step $i+1 \mapsto i+2$. $x_{i+1,i+2}$ is the cointersection of all coinserters coins$(x_{j+1,i+1} \cdot (f /\!\!/ h), g)$, for $j \leq i$, j even, and $x_{j,i+1} \cdot f \leq g \cdot h$:



If the Kan-Injective Reflection Chain

$$X = X_0 \xrightarrow{x_{01}} X_1 \xrightarrow{x_{12}} X_2 \longrightarrow \dots X_i \longrightarrow \cdots$$

converges at some even ordinal k (that is, $x_{k,k+2}$ is an isomorphism), then

$$X \xrightarrow{x_{0k}} X_k$$

is a reflection of X into LInj $\mathcal{H}$.

Let $\mathcal{X}$ be a poset enriched category with weighted colimits and a factorisation system $(\mathcal{E}, \mathcal{M})$ such that $\mathcal{E} \subseteq Epi(\mathcal{X})$, $\mathcal{M} \subseteq OrderMono(\mathcal{X})$, and $\mathcal{X}$ is $\mathcal{E}$-cowellpowered.

We say that $\mathcal{X}$ is locally ranked if, in addition, every object X of $\mathcal{X}$ has rank λ, for some regular cardinal λ; that is, the hom-functor hom$(X, _)$ preserves λ-directed unions of monomorphisms of $\mathcal{M}$.

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> For every set $\mathcal{H}$ of a locally ranked poset enriched category, the Kan-injective Subcategory Problem has an affirmative answer, that is, LInj $\mathcal{H}$ is reflective.

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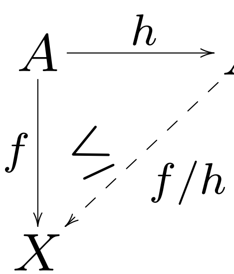
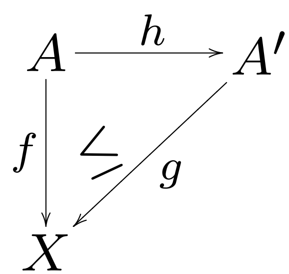
> For every set $\mathcal{H}$ of a locally ranked poset enriched category, the Kan-injective Subcategory Problem has an affirmative answer, that is, $\mathsf{LInj}\,\mathcal{H}$ is the Eilenberg-Moore category of a KZ-monad over the category.

Weak left Kan-injectivity

Given $h : A \to A'$,

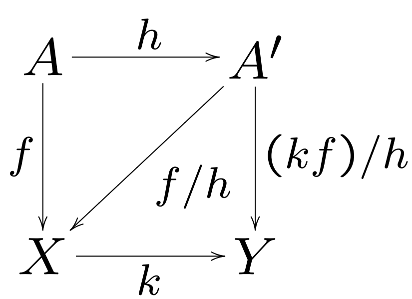
X is said to be weakly left Kan-injective w.r.t. h, if every $f : A \to X$ has a left Kan-extension:

(1) 

and (2) If 

then $f/h \leq g$.

$k : X \to Y$ is said to be weakly left Kan-injective w.r.t. h, if it preserves left Kan extensions, i.e., $(kf)/h = k(f/h)$ (with X and Y w. l. K. inj.)



$\mathsf{LInj}_w\mathcal{H}$:= subcategory of all objects and morphisms weakly left Kan injective w.r.t. $\mathcal{H}$

In every locally ranked poset enriched category, given a set $\mathcal{H}$ of morphisms there exists a class $\overline{\mathcal{H}}$ of morphisms with

$$\mathsf{LInj}_w\mathcal{H} = \mathsf{LInj}\,\overline{\mathcal{H}}.$$