Weights for The Object of Monoids

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Motivation

- Let T be a monad in a 2-category $\mathcal{K}$. There exists a notion of Eilenberg-Moore object for T[Street,1972].

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Question

How about '2-algebraic set' of monoids for any monoidal category object?

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Example

Starting with abelian groups in the category **Set**. Define an abelian group in the functor category $\mathbf{Set}^{\mathcal{C}^{\mathbf{op}}}$. Reflect by Yoneda embedding

$$Y_{\mathcal{C}} : \mathcal{C} \longrightarrow \mathbf{Set}^{\mathcal{C}^{\mathbf{op}}}$$

to derive notion of an abelian group in $\mathcal{C}$

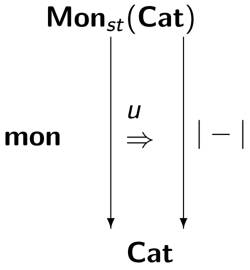
Description of procedure

- Start with a notion in **Cat**.
- Generalize in the pointwise manner to 2-functor categories $\mathbf{Cat}^{\mathcal{K}^{op}}$.
- Then using 2-Yoneda embedding

$$Y_{\mathcal{K}} : \mathcal{K} \longrightarrow \mathbf{Cat}^{\mathcal{K}^{op}}$$

one can reflect the algebraic notion in question to $\mathcal{K}$.

Objects of Monoids



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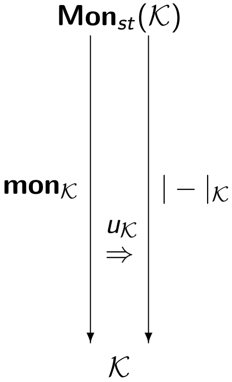
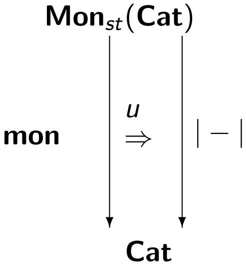
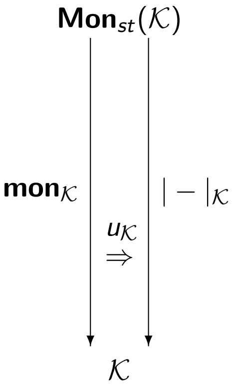


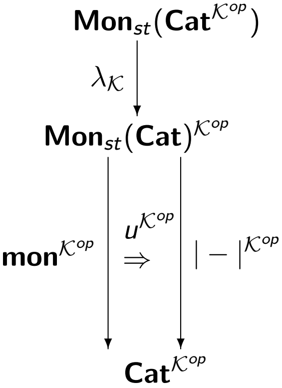


Objects of Monoids

$$\mathbf{mon}_{\mathcal{K}} \overset{u_{\mathcal{K}}}{\Rightarrow} |-|_{\mathcal{K}} : \mathbf{Mon}_{st}(\mathcal{K}) \longrightarrow \mathcal{K}$$

$$\mathbf{mon}^{\mathcal{K}^{op}} \overset{u^{\mathcal{K}^{op}}}{\Rightarrow} |-|^{\mathcal{K}^{op}} : \mathbf{Mon}_{st}(\mathbf{Cat})^{\mathcal{K}^{op}} \longrightarrow \mathbf{Cat}^{\mathcal{K}^{op}}$$


$\mathbf{Mon}_{st}(\mathcal{K})$
$\mathbf{mon}_{\mathcal{K}}$
$u_{\mathcal{K}}$
$\Rightarrow$
$|-|_{\mathcal{K}}$
$\mathcal{K}$


$\mathbf{Mon}_{st}(\mathbf{Cat}^{\mathcal{K}^{op}})$
$\lambda_{\mathcal{K}}$
$\mathbf{Mon}_{st}(\mathbf{Cat})^{\mathcal{K}^{op}}$
$\mathbf{mon}^{\mathcal{K}^{op}}$
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$\Rightarrow$
$|-|^{\mathcal{K}^{op}}$
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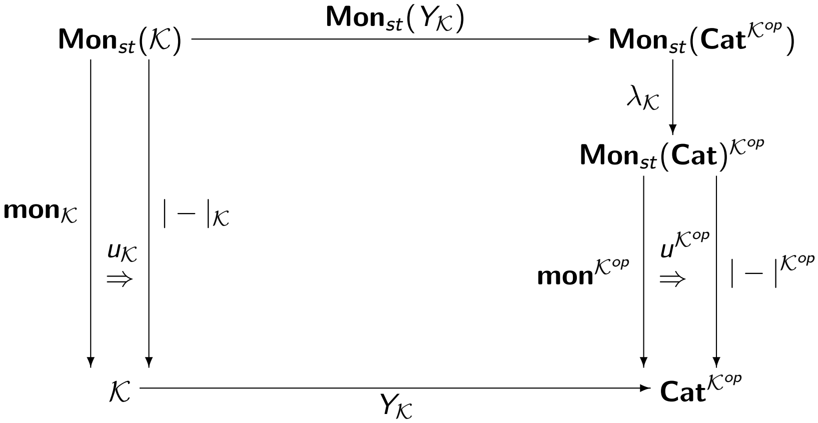

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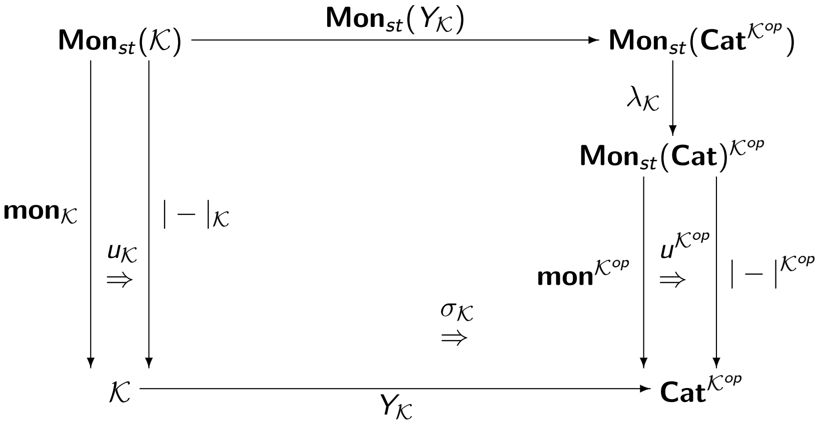

Monadicity of the object of monoids

Theorem (Monadicity for the monoids)

Let $(\mathcal{C}, \otimes, I)$ *be a monoidal category and* $U : \mathbf{mon}(\mathcal{C}) \to \mathcal{C}$ *be its category of monoids together with the canonical forgetful functor* U*. Suppose that* U *is a right adjoint. Then* U *is monadic i.e* $\mathbf{mon}(\mathcal{C})$ *is the Eilenberg-Moore object for the corresponding monad.*

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Theorem (S, Zawadowski)

Let $(c, \otimes, I)$ *be a monoidal object in a 2-category* $\mathcal{K}$ *and* $u : \mathbf{mon}(c) \to c$ *be the object of monoids for* $(c, \otimes, I)$*. Suppose that* u *is a right adjoint. Then* u *is monadic i.e* $\mathbf{mon}(c)$ *is the Eilenberg-Moore object for the corresponding monad.*

Motivation

- Let T be a monad in a 2-category $\mathcal{K}$. Then the Eilenberg-Moore object for T is a weighted 2-limit [Lawvere,1969].

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Question

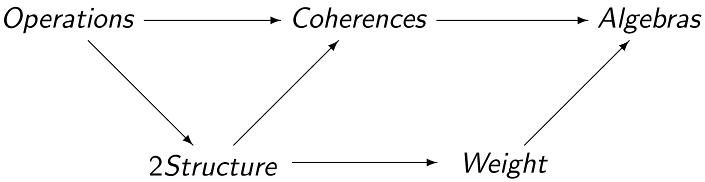
Is '2-algebraic set' of monoids a weighted 2-limit for a suitable diagram?

The weights

- Three 2-categories:

$$\mathit{Operations} \longrightarrow \mathit{Coherences} \longrightarrow \mathit{Algebras}$$

- *iso on category part/locally fully faithful* factorizations



The weights

- 2-category $2\mathit{Structure}$ will be a 2-Lawvere theory.
- We have a 2-functor

$$2\mathit{Structure} \longrightarrow \mathit{Weight}$$

Weight has the terminal object 1

$$2\mathit{Structure} \longrightarrow \mathit{Weight} \xrightarrow{\mathit{Weight}(1,-)} \mathbf{Cat}$$

is the weight $\mathcal{W}$.

- Let

$$F : 2\mathit{Structure} \longrightarrow \mathcal{K}$$

be a finite products preserving 2-functor. Pick $\mathit{lim}_{\mathcal{W}} F$.

The weights

- To construct a sequence:

$$\textit{Operations} \longrightarrow \textit{Coherences} \longrightarrow \textit{Algebras}$$

  we will use symmetric operads.
- Define categories and 2-categories associated with symmetric operads: $\mathbb{F}_A$, $\left[\begin{array}{cc} A & B \end{array}\right]$, $\left[\begin{array}{cc} A_1 & B_1 \\ A_0 & B_0 \end{array}\right]$.

The Category $\mathbb{F}_A$

- A a symmetric operad with multiplication μ^A.
- The objects of $\mathbb{F}_A$ are $[n] = \{1, ..., n\}$ for $n \in \mathbb{N}$.
- Morphims are

$$[n] \xrightarrow{\langle f, a_i \rangle_{i \in [m]}} [m]$$

where $a_i \in A(f^{-1}(i))$.

The Category $\mathbb{F}_A$

$$\begin{array}{c} [k] \\ \Big\downarrow {\scriptstyle (g,\, b_j)_{j \in [n]}} \\ [n] \\ \Big\downarrow {\scriptstyle (f,\, a_i)_{i \in [m]}} \\ [m] \end{array}$$

The Category $\mathbb{F}_A$

$$\begin{array}{c} g^{-1}(f^{-1}(i)) \\ \Big\downarrow{\scriptstyle g} \\ f^{-1}(i) \\ \Big\downarrow{\scriptstyle f} \\ i \end{array}$$

The Category $\mathbb{F}_A$

$$\begin{array}{ccc} & & g^{-1}(f^{-1}(i)) \\ & & \downarrow{\scriptstyle g} \\ (f^{-1}(i), a_i) & & f^{-1}(i) \\ \downarrow & & \downarrow{\scriptstyle f} \\ i & & i \end{array}$$

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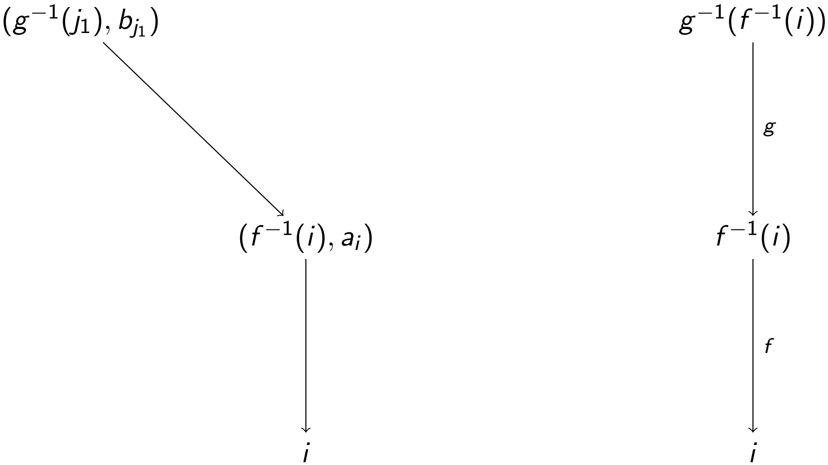

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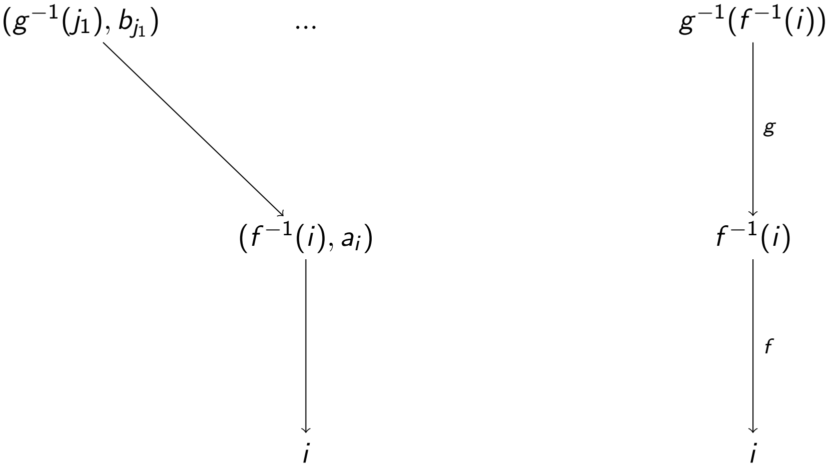

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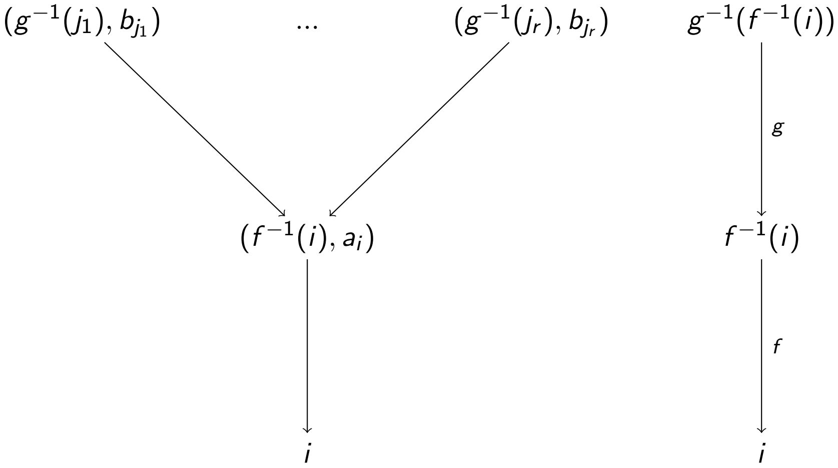

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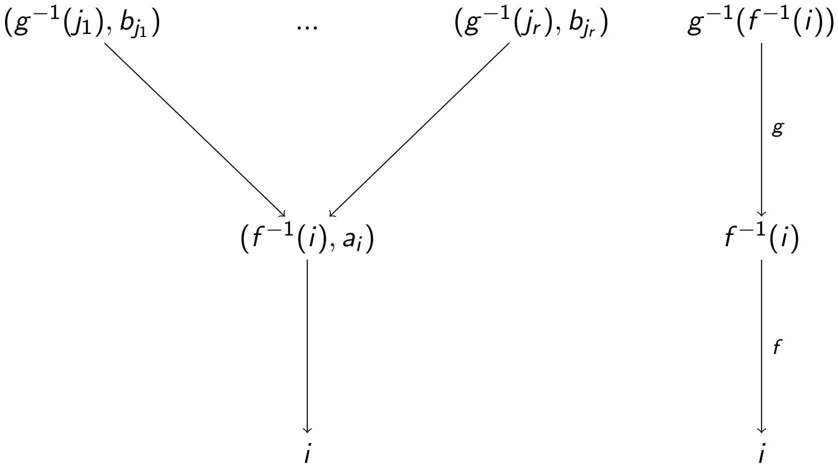


Use multiplication in A to get $c_i = \mu^A(b_{j_1}, ..., b_{j_r}, a_i)$. The composition is $(f \circ g, c_i)$

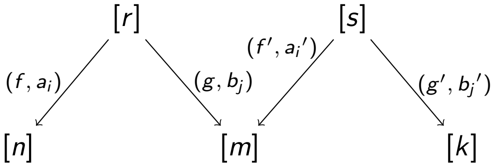
Categories of spans from symmetric operads $\left[\begin{array}{cc} A & B \end{array}\right]$

- The objects of $\left[\begin{array}{cc} A & B \end{array}\right]$ are $[n]$.
- Morphism of $\left[\begin{array}{cc} A & B \end{array}\right]$ are classes

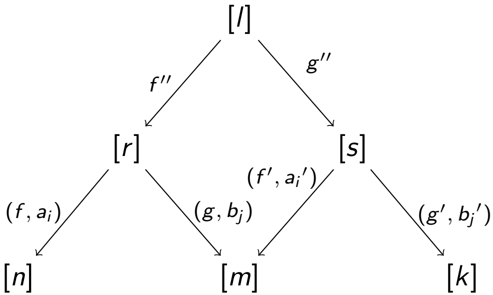
$$\begin{array}{ccc} & [r] & \\ {\scriptstyle \langle f, a_i \rangle_{i \in [n]}} \swarrow & & \searrow {\scriptstyle \langle g, b_j \rangle_{j \in [m]}} \\ [n] & & [m] \end{array}$$

with $\langle f, a_i \rangle_{i \in [n]}$ in $\mathbb{F}_A$ and $\langle g, b_j \rangle_{j \in [m]}$ in $\mathbb{F}_B$.

Categories of spans from symmetric operads $\left[\ A \quad B\ \right]$



Categories of spans from symmetric operads [A B]



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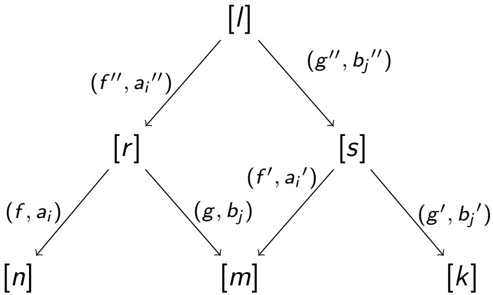


Arrays of symmetric operads and higher categories of spans

- $\alpha_0 : A_1 \to A_0$, $\alpha_1 : A_1 \to B_0$, $\beta_0 : B_1 \to A_0$, $\beta_1 : B_1 \to B_0$ morphisms of symmetric operads.
- 2-category of 2-spans defined by an array of symmetric operads $\begin{bmatrix} A_1 & B_1 \\ A_0 & B_0 \end{bmatrix}$
- Its category part is $\begin{bmatrix} A_0 & B_0 \end{bmatrix}$.
- Its 2-cells are 2-spans defined using A_1 and B_1 and morphisms given above.

Some Operads

- *BTr* operad of binary trees i.e BTr(X) is the set of all binary trees with leaves partially labeled by elements of X
- *Lo* operad of linear orders i.e Lo(X) is the set of all linear orders on X
- Have a morphism

$$BTr \to Lo$$

  that remembers order of leaves.
- $\bot$,$\top$ the initial and the terminal symmetric operads, respectively.
- Have a sequence of symmetric operads

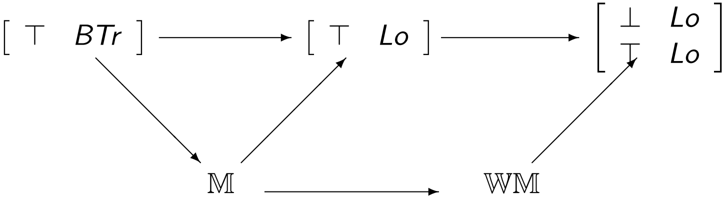
$$\bot \to BTr \to Lo \to \top$$

The weights for the objects of monoids

- Consider the following two 2-functors

$$\left[\ \top \quad BTr\ \right] \longrightarrow \left[\ \top \quad Lo\ \right] \longrightarrow \left[\begin{array}{cc} \bot & Lo \\ \top & Lo \end{array}\right]$$

- Take a factorization (iso on category part/locally fully faithful):



The weights for the objects of monoids

Theorem (S. Zawadowski)

The 2-category $\mathbb{M}$ is the 2-Lawvere theory for the monoidal category objects. The 2-functor $\mathcal{W}$, the composite of

$$\mathbb{M} \longrightarrow \mathbb{WM} \xrightarrow{\quad \mathbb{WM}(1,-) \quad} \mathbf{Cat}$$

is the weight for objects monoids over a monoidal category objects i.e., a finite product preserving 2-functor $F : \mathbb{M} \to \mathcal{K}$ corresponds to a monoidal category object in $\mathcal{K}$ and, if it exists, the weighted limit $lim_{\mathcal{W}}F$ is the object of monoids for F in $\mathcal{K}$.

The weights for other sturctures

Comonoids

$\left[\begin{array}{cc} \top & BTr \end{array}\right] \longrightarrow \left[\begin{array}{cc} \top & Lo \end{array}\right] \longrightarrow \left[\begin{array}{cc} Lo & \bot \\ \top & Lo \end{array}\right]$ the weight for the object of comonoids over a monoidal category object.

Commutative monoids

$\left[\begin{array}{cc} \top & BTr \end{array}\right] \longrightarrow \left[\begin{array}{cc} \top & \top \end{array}\right] \longrightarrow \left[\begin{array}{cc} \bot & \top \\ \top & \top \end{array}\right]$ the weight for the object of commutative monoids over a symmetric monoidal category object.

Bi-monoids

$\left[\begin{array}{cc} \top & BTr \end{array}\right] \longrightarrow \left[\begin{array}{cc} \top & \top \end{array}\right] \longrightarrow \left[\begin{array}{cc} Lo & Lo \\ \top & \top \end{array}\right]$ the weight for the object of bi-monoids over a symmetric monoidal category object.

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- Microcosm prinicple must be formulated carefully.
- Using colored symmetric operads one can construct the weights for the objects of actions.
- As in **Cat** lax monoidal 1-cells induce 1-cells between the objects of monoids

The End