On a fat small object argument

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Theorem. (Joyal, Wraith 1984) Any acyclic simplicial set is a filtered colimit of finitely presentable acyclic simplicial sets.

This result was a key step in their proof that Eilenberg-Mac Lane toposes and Eilenberg-Mac Lane spaces have the same cohomology.

Theorem 1. (Raptis, JR) The category $\mathcal{W}$ of weak equivalences of simplicial sets is finitely accessible.

More precisely

(1) $\mathcal{W}$ is closed in $\mathbf{SSet}^{\to}$ under filtered colimits,

(2) any $f \in \mathcal{W}$ is a filtered colimit of weak equivalences between finitely presentable simplicial sets.

(1) is well known and it is not difficult to show that any $f \in \mathcal{W}$ is a filtered colimit of weak equivalences between countably presentable simplicial sets.

(2) is based on a *fat small object argument*.

A *finitely combinatorial category* is a locally finitely presentable category $\mathcal{K}$ equipped with a set $\mathcal{X}$ of morphisms between finitely presentable objects.

Cellular morphisms are transfinite compositions of pushouts of morphisms from $\mathcal{X}$ and *cofibrations* are retracts of cellular morphisms.

An object K is *cofibrant* if the unique morphism $0 \to K$ from the initial object is a cofibration.

Theorem 2. (Makkai, JR, Vokřínek) In a finitely combinatorial category, every cofibrant object is a filtered colimit of finitely presentable cofibrant objects.

Let κ be a regular cardinal. One can define κ-combinatorial categories and one has the analogy of Theorem 2 for them.

A category is *combinatorial* if it is κ-combinatorial for some κ.

Corollary 1. Every trivial cofibration in **SSet** is a filtered colimit of trivial cofibrations between finitely presentable objects.

Trivial cofibrations are cofibrant objects in $\mathcal{K} = \mathbf{SSet}^{\to}$ and $\mathcal{X} = \mathcal{X}_1 \cup \mathcal{X}_2$ where

$$\mathcal{X}_1 = \{(i, i) : (\mathrm{id} : \partial\Delta_n \to \partial\Delta_n) \to (\mathrm{id} : \Delta_n \to \Delta_n) | i \text{ boundary inclusion}\}$$

$$\mathcal{X}_2 = \{(\mathrm{id}, j) : (\mathrm{id} : \Lambda_n^k \to \Lambda_n^k) \to (j : \Lambda_n^k \to \Delta_n) | j \text{ horn inclusion}\}.$$

The proofs of Theorems 1 and 2 use *good colimits* introduced by Lurie.

Theorem 1 can be proved for any finitely combinatorial model category on a Grothendieck topos where cofibrations are monomorphisms and a cylinder functor preserves finitely presentable objects.

A poset P is good if it is well-founded and has the least element $\bot$.

Well-ordered sets and shape posets for pushouts are good.

An element $x \in P$ is *isolated* if there is a top element x^- strictly below x.

A non-isolated element distinct from $\bot$ is called *limit*.

A *good* diagram $D : P \to \mathcal{K}$ is such that Dx is a colimit of the restriction of D on elements strictly below x for each limit x.

The *composition* of D is the component $\delta_\bot$ of the colimit cocone.

Links of D are morphisms $D(x^- \to x)$ for x isolated.

Proposition 1. (Lurie) Let $\mathcal{X}$ be a class of morphisms in a cocomplete category $\mathcal{K}$. Then the composition of a good diagram with links in $\mathrm{Po}(\mathcal{X})$ is cellular.

$\mathrm{Po}(\mathcal{X})$ consists of pushouts of morphisms from $\mathcal{X}$.

A good poset is κ-*good* if all its initial segments $\downarrow x$ have cardinality $< \kappa$.

Theorem 3. (Makkai, JR, Vokřínek) Let $\mathcal{K}$ be a cocomplete category, κ a regular cardinal and $\mathcal{X}$ a class of morphisms with κ-presentable domains. Then every cellular morphism is a composition of a κ-good κ-directed diagram with links in $\mathrm{Po}(\mathcal{X})$.

This result may be called a *fat small object argument* because it replaces a thin transfinite composition containing large objects by a fat good composition of small objects.

Corollary 2. Let $\mathcal{K}$ be a cocomplete category, κ a regular cardinal and $\mathcal{X}$ a class of morphisms between κ-presentable objects. Then any cofibrant object is a κ-filtered colimit of κ-presentable cofibrant objects.

This follows from Theorem 3 for cellular objects and the passage to cofibrant objects uses [Makkai, Paré].

Theorem 4. (Makkai, JR, Vokřínek) Let $\mathcal{K}$ be a κ-combinatorial category where κ is uncountable. Then any cofibrant object is a κ-good, κ-directed colimit of κ-presentable cofibrant objects where all links are cofibrations.

This follows from Theorem 3 as well but, instead of treating retracts via [Makkai, Paré], we use [Lurie]:

$$\text{retracts of cellular} = \text{cellular of retracts.}$$

Lurie introduced good colimits for proving this result. Independently, one used generalized Hill lemma for the same purpose in the additive context.

Let $\mathcal{K}$ be the category of R-modules and $\mathcal{X} = \{0 \to R\}$. This is a finitely combinatorial category whose cellular objects are free modules and cofibrant objects are projective modules. Theorem 4 implies the classical theorem of Kaplansky: every projective module is a coproduct of countably generated projective modules. This also shows that Theorem 4 is not valid for $\aleph_0$.

A functor $F : \mathcal{K} \to \mathcal{L}$ between combinatorial categories is called *combinatorial* if it preserves colimits and cofibrations.

Any combinatorial functor is a left adjoint.

COMB will denote the 2-category of combinatorial categories, combinatorial functors and natural transformations.

COMB is legitimate but not locally small and is equipped with the 2-functor $U : \mathbf{COMB} \to \mathbf{LOC}$ to the 2-category **LOC** of locally presentable categories, colimit preserving functors and natural transformations.

The 2-category **LOC** has all PIE-limits, i.e., products, inserters and equifiers (Makkai, Paré). Consequently, its has all lax limits and all pseudolimits. All these limits are calculated in **CAT**.

Theorem 5. (Makkai, JR) **COMB** has PIE-limits preserved by U.

Corollary 3. (Lurie) Let $\mathcal{K}$ be a combinatorial category and $\mathcal{C}$ a small category. Then $\mathcal{K}^{\mathcal{C}}$ is combinatorial (with respect to pointwise cofibrations).

Let **CMOD** denote the 2-category of combinatorial model categories, left Quillen functors and natural transformations.

CMOD is legitimate but not locally small and is equipped with the 2-functors $V_1, V_2 : \mathbf{CMOD} \to \mathbf{COMB}$, $V_1(\mathcal{K}) = (\mathcal{K}, \mathcal{C})$ and $V_2(\mathcal{K}) = (\mathcal{K}, \mathcal{C}_0)$ where $\mathcal{C}$ is the class of cofibrations and $\mathcal{C}_0$ is the class of trivial cofibrations.

Theorem 6. (Barwick) **CMOD** has lax limits preserved by $V_1, V_2 : \mathbf{CMOD} \to \mathbf{COMB}$.

Problem. Does **CMOD** have PIE-limits? Equivalently, does **CMOD** have pseudopullbacks?

Proposition 2. V_2 preserves PIE-limits (pseudopullbacks).

V_2 has a left adjoint sending $(\mathcal{K}, \mathcal{X})$ to $(\mathcal{K}, \mathcal{X}, \mathcal{X})$.

We have an example of a pseudopullback in **CMOD** which is not preserved by V_1.

Let $\mathcal{K}$ be the standard model category of simplicial sets. Let $t : 0 \to 1$ and $\mathcal{L}$ be the model structure on simplicial sets where $\mathrm{cof}(\{t\})$ is the class of cofibrations and any morphism is a weak equivalence. Let $\mathcal{K}_t$ be the trivial model structure in $\mathcal{K}$ (any morphism is a trivial cofibration) and $\mathcal{K}_d$ be the discrete model structure (any morphism is a trivial fibration). Then the intersection of $\mathcal{K}$ and $\mathcal{L}$ over $\mathcal{K}_t$ is $\mathcal{K}_d$. Since $V_1(\mathcal{L}) \subseteq V_1(\mathcal{K})$, this intersection is not preserved by V_1.