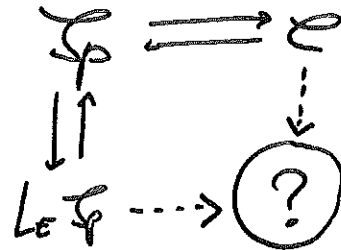


HOMOLOGICAL LOCALISATION OF MODEL CATEGORIES Eilenberg 100, 40 mins

[j.t.w. D. Barnes]

Idea: study spectra via Bousfield localisation w.r.t. homology theories E_*
study spectra by relating them to other model categories

Goal: "E-localise" arbitrary \mathcal{C} :



(1) Bousfield localisation

create model str. $L_E \mathcal{S}p$ on spectra s.th. the weak equivalences are E_* -isomorphisms (rather than \mathbb{Z}_n -isos)

\mathcal{C} model category (stable), S class of maps
 $Z \in \mathcal{C}$ is S-local if $[B, Z]^{\mathcal{C}} \rightarrow [A, Z]^{\mathcal{C}}$ is an iso $\forall A \rightarrow B$ in S .
 $f: X \rightarrow Y$ is an S-equivalence if $[Y, Z]^{\mathcal{C}} \rightarrow [X, Z]^{\mathcal{C}}$ is an iso $\forall S$ -local Z .

N.B: $S \not\subseteq$ S-equivalences

Theorem [Hirschhorn]

\mathcal{C} "nice", S set \Rightarrow there is a model str. $L_S \mathcal{C}$ on \mathcal{C} s.th.

- cofibrations in $L_S \mathcal{C} =$ cofibs in \mathcal{C}
- weak equiv. = S-equiv.

[EKMM] There is a set J_E s.th.

J_E -equiv. = E_* -isos of spectra

(2) Framings

\mathcal{C} stable model cat, $X \in \mathcal{C}$ fibrant + cofibrant

[Lenhardt] There is a Quillen pair $F: \mathcal{I}_p \rightleftarrows \mathcal{C} : G$ with $F(S) \cong X$.

- Notation: $(F, G) = (X \perp -, \text{Hom}(X, -))$
- this pair is unique up to htpy
- every Quillen functor $F: \mathcal{I}_p \rightarrow \mathcal{C}$ is of the form $X \perp -$
- this makes $\text{Ho}(\mathcal{C})$ a $\text{Ho}(\mathcal{I}_p)$ -module category.

(3) E-localisation

First idea: (bad) Define $f: X \rightarrow Y$ in \mathcal{C} to be a "naive E-equiv." if $X \wedge E \xrightarrow{f \wedge E} Y \wedge E$ is an iso in $\text{Ho}(\mathcal{C})$.
 This localisation would lead to self-theoretical problems!

Theorem [B-R] \mathcal{C} stable and "nice"

I_E generating cofibrations

J_E as before

$$S := I_E \square J_E$$

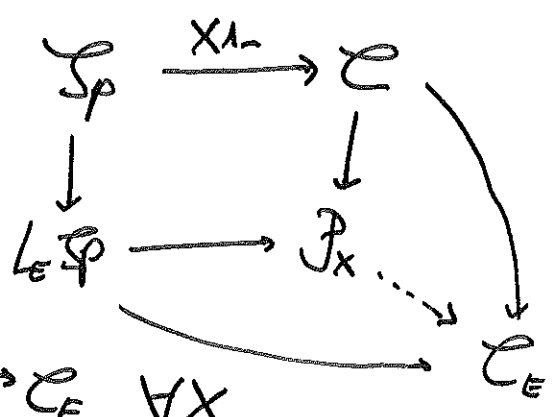
\nwarrow pushout-prod., defined via framings

Then $\mathcal{C}_E := L_S \mathcal{C}$ satisfies the following:

- every Quillen pair $\mathcal{I}_p \rightleftarrows \mathcal{C}$ factors as
- $$\begin{array}{ccc} \mathcal{I}_p & \rightleftarrows & \mathcal{C} \\ \downarrow \uparrow & & \downarrow \uparrow \\ L_S \mathcal{I}_p & \rightleftarrows & \mathcal{C}_E \end{array}$$

- \mathcal{C}_E is the "closest" model cat. to \mathcal{C} with this property

- For every X , get homotopy pushout \mathcal{P}_X (\mathcal{C} with another model str.)



optional, see time

\Rightarrow get Quillen pairs $\mathcal{P}_X \rightleftarrows \mathcal{C}_E \quad \forall X$
 \mathcal{C}_E is the closest model str. to all the \mathcal{P}_X .

Examples and properties

- $\mathcal{C} = R\text{-mod}$ for R ring spectrum $\Rightarrow \mathcal{C}_E = L_E(R\text{-mod})$
- $\text{RHom}_{\mathcal{C}_E}(X, Y) = \text{RHom}_E(X, Y_E)$ is an E -local spectrum
↑
fibrant replacement in \mathcal{C}_E
- $E(n)$ Johnson-Wilson theories
 $K(n)$ Morava-K-theories
 \mathcal{C} algebraic model cat. $\Rightarrow \text{RHom}_E(X, Y)$ is an ETL-spectrum $\forall X, Y$
 $\Rightarrow \text{RHom}_{\mathcal{C}_E}(X, Y)$ ETL and E -local
- [Guhmura?] $L_{K(n)} HG \simeq * \quad (n \geq 1) \Rightarrow \mathcal{C}_{K(n)}$ is trivial
 $L_{E(n)} HG \simeq HG \wedge HQ \Rightarrow \mathcal{C}_{E(n)} = \mathcal{C}_{HQ}$
- $\Rightarrow L_{E(n)} \mathcal{P}$ is only Quillen equiv. to something algebraic if $n=0$
 $L_{K(n)} \mathcal{P}$ is never Quillen equiv. to an algebraic model cat.