HOMOLOGICAL LOCALISATION OF MODEL CATEGORIES

Eilenberg 100, 40 mins

[jt. w. D. Barnes]

Idea: study spectra via Bousfield localisation w.r.t. homology theories E_*

study spectra by relating them to other model categories

Goal: "E-localise" arbitrary $\mathcal{C}$:

$$\begin{tikzcd} Sp \arrow[r, shift left] \arrow[d, shift right] & \mathcal{C} \arrow[l, shift left] \arrow[d, dashed] \\ L_E Sp \arrow[u, shift right] \arrow[r, dashed] & ? \end{tikzcd}$$

(1) Bousfield localisation

create model str. $L_E Sp$ on spectra s.th. the weak equivalences are E_*-isomorphisms (rather than π_*-isos)

$\mathcal{C}$ model category (stable), S class of maps

$Z \in \mathcal{C}$ is S-local if $[B,Z]^{\mathcal{C}} \longrightarrow [A,Z]^{\mathcal{C}}$ is an iso $\forall A \to B$ in S.

$f: X \to Y$ is an S-equivalence if $[Y,Z]^{\mathcal{C}} \longrightarrow [X,Z]^{\mathcal{C}}$ is an iso $\forall$ S-local Z.

N.B: $S \subsetneq$ S-equivalences

Theorem [Hirschhorn]

$\mathcal{C}$ "nice", S set $\Rightarrow$ there is a model str. $L_S\mathcal{C}$ on $\mathcal{C}$ s.th.

- cofibrations in $L_S\mathcal{C}$ = cofibs in $\mathcal{C}$
- weak equiv. = S-equiv.

[EKMM] There is a set J_E s.th.

J_E-equiv. = E_*-isos of spectra

(2) Framings

$\mathcal{C}$ stable model cat., $X \in \mathcal{C}$ fibrant + cofibrant

[Lenhardt] There is a Quillen pair $F: \mathcal{S}p \rightleftarrows \mathcal{C} : G$
with $F(\mathbb{S}) \cong X$.

- Notation: $(F, G) = (X \wedge -, \mathrm{Hom}(X, -))$
- this pair is unique up to htpy
- every Quillen functor $F: \mathcal{S}p \longrightarrow \mathcal{C}$ is of the form $X \wedge -$
- this makes $Ho(\mathcal{C})$ a $Ho(\mathcal{S}p)$-module category.

(3) E-localisation

First idea: (bad) Define $f: X \longrightarrow Y$ in $\mathcal{C}$ to be a "naive E-equiv." if $X \wedge^L E \xrightarrow{f \wedge E} Y \wedge^L E$ is an iso in $Ho \mathcal{C}$.
This localisation would lead to set-theoretical problems!

Theorem [B-R] $\mathcal{C}$ stable and "nice"
$I_\mathcal{C}$ generating cofibrations
J_E as before
$S := I_\mathcal{C} \square J_E$
pushout-prod., defined via framings

Then $\mathcal{C}_E := L_S \mathcal{C}$ satisfies the following:

- every Quillen pair factors as

$$\begin{array}{ccc} \mathcal{S}p & \rightleftarrows & \mathcal{C} \\ \downarrow\uparrow & & \downarrow\uparrow \\ L_E \mathcal{S}p & \rightleftarrows & \mathcal{C}_E \end{array}$$

- $\mathcal{C}_E$ is the "closest" model cat. to $\mathcal{C}$ with this property

- For every X, get homotopy pushout $\mathcal{P}_X$ ($\mathcal{C}$ with another model str.)

$$\begin{tikzcd}
\mathcal{S}p \arrow[r, "X\wedge -"] \arrow[d] & \mathcal{C} \arrow[d] \arrow[ddr, bend left] & \\
L_E\mathcal{S}p \arrow[r] \arrow[drr, bend right] & \mathcal{P}_X \arrow[dr, dashed] & \\
 & & \mathcal{C}_E
\end{tikzcd}$$

$\Rightarrow$ get Quillen pairs $\mathcal{P}_X \rightleftarrows \mathcal{C}_E \ \forall X$

$\mathcal{C}_E$ is the closest model str. to all the $\mathcal{P}_X$.

optional, see time

<u>Examples and properties</u>

- $\mathcal{C} = R\text{-mod}$ for R ring spectrum $\Rightarrow \mathcal{C}_E = L_E(R\text{-mod})$
- $RHom_{\mathcal{C}_E}(X,Y) = RHom_{\mathcal{C}}(X, Y_E)$ is an E-local spectrum

  fibrant replacement in $\mathcal{C}_E$

- $E(n)$ Johnson-Wilson theories

  $K(n)$ Morava-K-theories

  $\mathcal{C}$ algebraic model cat. $\Rightarrow RHom_{\mathcal{C}}(X,Y)$ is an EML-spectrum $\forall X, Y$

  $\Rightarrow RHom_{\mathcal{C}_E}(X,Y)$ EML and E-local

  [Gutiérrez] $L_{K(n)}HG \simeq * \ (n \geq 1) \Rightarrow \mathcal{C}_{K(n)}$ is trivial

  $L_{E(n)}HG \simeq HG \wedge HQ \Rightarrow \mathcal{C}_{E(n)} = \mathcal{C}_{HQ}$

  $\Rightarrow L_{E(n)}\mathcal{S}p$ is only Quillen equiv. to something algebraic if $n=0$

  $L_{K(n)}\mathcal{S}p$ is never Quillen equiv. to an algebraic model cat.