Full and almost full embeddings of the Graphs into the Abelian Groups and other categories – motivated by localizations

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Samuel Eilenberg Centenary, 2013

Plan:

- Localizations
- Orthogonal Subcategory Problem
- (weak) Vopěnka principle 196? (1988)
- (weak) Vopěnka implies ...
- negation of (weak) Vopěnka implies ...
  - Embeddings of **Graphs** into ... 1969-1980 and 2010- ...

Orthogonality pairs

We say that a morphism $f : A \to B$ is orthogonal to an object Z, and write $f \perp Z$, if f induces a bijection:

$$f^* : \mathrm{Hom}(B, Z) \xrightarrow{\cong} \mathrm{Hom}(A, Z)$$

Given a class of morphisms $\mathcal{E}$ and a class of objects $\mathcal{D}$, a pair $(\mathcal{E}, \mathcal{D})$ is an orthogonality pair if $\mathcal{E} = {}^{\perp}\mathcal{D}$ and $\mathcal{D} = \mathcal{E}^{\perp}$.

- $\mathcal{E}$ is closed under colimits.
- $\mathcal{D}$ is closed under limits.

Localizations $L : \mathcal{C} \to \mathcal{C}$

A localization is a functor $L : \mathcal{C} \to \mathcal{C}$ with a natural transformation (coaugmentation) $\eta : Id \to L$ such that

1. $L\eta_X : LX \to LLX$ is an isomorphism for all X
2. For all X and Y in $\mathcal{C}$ we have $\eta_X \perp LY$ that is:

$$\eta_X^* : \mathrm{Hom}(LX, LY) \xrightarrow{\cong} \mathrm{Hom}(X, LY)$$

Localizations may be viewed as projections
onto the class of local objects $\mathcal{D} = \{Z \mid \eta_Z : Z \xrightarrow{\cong} LZ\}$
along the class of L-equivalences $\mathcal{E} = \{f \mid Lf \text{ is an isomorphism}\}$

For every localization L the pair $(\mathcal{E}, \mathcal{D})$ above is an orthogonality pair.

Orthogonal subcategory problem:

A subcategory $\mathcal{D}$ is called reflective if $({}^{\perp}\mathcal{D}, \mathcal{D})$ is associated with some localization.

Is every full, closed under limits subcategory $\mathcal{D}$ reflective?

The answer depends on set theory:

in **Graphs**:

- **NO** is consistent with ZFC
- weak Vopěnka's principle is equivalent to **YES** (Adámek, Rosický, Trnková 1988)

Vopěnka implies localizations in other categories

Theorem (Adámek, Rosický, Trnková 1988)

Weak Vopěnka principle implies that every full, closed under limits subcategory of a locally presentable category is reflective.

A choice of locally presentable categories:

- category of groups
- category of fields
- category of R-modules
- category of Hilbert spaces
- category of partially ordered sets
- category of simplicial sets
- category of metrizable spaces and continuous maps
- category of CW-complexes and continuous maps
- category of models of some first order theory
- many more

Tool: Adámek, Rosický Locally presentable and accessible categories, Theorem 2.65.

. . . in not locally presentable categories

Theorem (Casacuberta, Scevenels, Smith 2005)

Assuming Vopěnka principle every orthogonality pair $(\mathcal{E}, \mathcal{D})$ *in the homotopy category is associated with some localization.*

Theorem (Casacuberta, Gutiérrez, Rosický 2011 preprint)

Assuming Vopěnka principle every orthogonality pair $(\mathcal{E}, \mathcal{D})$ *in a stable homotopy category is associated with some localization.*

Negation of (weak) Vopěnka implies ...

Theorem (Definition)

The negation of Vopěnka principle is equivalent to each of the following:

- **Graphs** *contains a large rigid class of objects.*
- **Ord** *can be fully embedded in* **Graphs**.

The negation of weak Vopěnka principle is equivalent to the following:

- $\mathbf{Ord}^{\mathrm{op}}$ *can be fully embedded in* **Graphs**.

Having a full embedding of **Ord** or $\mathbf{Ord}^{op}$ into a category $\mathcal{C}$ usually allows to construct a nonreflexive subcategory of $\mathcal{C}$.

Graphs admits a full embedding into some categories:

- Category of semigroups (Hedrlín, Lambek 1969)
- Category of integral domains (Fried, Sichler 1977)
- Category of posets (Pultr, Trnková 1980)

And almost full – up to constant maps – embedding into some more categories:

- Category of metric spaces (Trnková, 1972)
- Category of paracompact spaces (Koubek, 1974)

But more general almost full functors **Graphs** $\longrightarrow \mathcal{C}$ also allow to construct nonreflective subcategories.

Theorem (AP 2010)

There exists an embedding

$$F : \mathbf{Graphs} \longrightarrow \mathbf{Groups}$$

which induces natural bijections:

$$\mathrm{Hom}_{\mathbf{Graphs}}(X, Y) \cup \{*\} \xrightarrow{\cong} \mathrm{Rep}(FX, FY) = \mathrm{Hom}_{\mathbf{Groups}}(FX, FY)/FY$$

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Theorem (AP 2011 preprint)

There exists an embedding

$$G : \textbf{Graphs} \longrightarrow \textbf{AbelianGroups}$$

which induces natural isomorphisms:

$$\mathbb{Z}[\mathrm{Hom}_{\textbf{Graphs}}(X, Y)] \xrightarrow{\cong} \mathrm{Hom}_{\mathbb{Z}}(GX, GY)$$

Corollary (Isbell question 1966)

Negation of weak Vopěnka principle implies the existence of a full, closed under limits, nonreflective subcategory of **AbelianGroups**.

Idea of proof

$\Gamma \subseteq$ **Graphs** full subcategory of countable graphs.
$A = \mathbb{Z}[\Gamma]$ "category ring".
$A \cong \mathrm{Hom}_{\mathbb{Z}}(M, M)$, $\quad \Gamma \subseteq A \subseteq M$. *(Corner ...)*
$M \cong M \cdot \mathrm{id}_C \oplus M \cdot (1 - \mathrm{id}_C)$
$\mathbb{Z}[\mathrm{Hom}_{\mathbf{Graphs}}(C, D)] \xrightarrow{\cong} \mathrm{Hom}_{\mathbb{Z}}(GC, GD)$
Define G : **Graphs** $\longrightarrow$ **AbelianGroups**

$$GX = \operatorname*{colim}_{\substack{\{C \to X\} \\ C \in \Gamma}} M \cdot \mathrm{id}_C$$

Then prove:

$$\mathbb{Z}[\mathrm{Hom}(\mathrm{colim}\, C_i, \mathrm{colim}\, D_j)] \cong \lim \mathrm{colim}\, \mathbb{Z}[\mathrm{Hom}(C_i, D_j)] \cong$$

$$\cong \lim \mathrm{colim}\, \mathrm{Hom}_{\mathbb{Z}}(GC_i, GD_j) \cong \mathrm{Hom}_{\mathbb{Z}}(\mathrm{colim}\, GC_i, \mathrm{colim}\, GD_j)$$

Theorem (Göbel, AP 2013)

For every cotorsion free ring R, there exists an embedding

$$G : \mathbf{Graphs} \longrightarrow \mathbf{Mod}_R$$

which induces natural isomorphisms:

$$R[\mathrm{Hom}_{\mathbf{Graphs}}(X, Y)] \xrightarrow{\cong} \mathrm{Hom}_R(GX, GY)$$

work in progress...

Theorem (AP)

There exists an embedding

$$E : \mathbf{Graphs} \longrightarrow \mathbf{Rings}$$

which induces natural isomorphisms:

$$\mathrm{Hom}_{\mathbf{Graphs}}(X, Y) \xrightarrow{\cong} \mathrm{Hom}_{\mathbf{Rings}}(EX, EY)$$

$$EY[\mathrm{Hom}_{\mathbf{Graphs}}(X, Y)] \xrightarrow{\cong} \mathrm{Hom}_{\mathbb{Z}}(EX, EY)$$

where $R[H] = \{\sum hr_h \mid h \in H, r_h \in R\}$

and $hr \cdot nq = hn \cdot (r)nq$

One thing to remember...

The category of Abelian Groups is as complicated as **Graphs** or any other point set category:

$$G : \textbf{Graphs} \longrightarrow \textbf{AbelianGroups}$$

$$\mathbb{Z}[\mathrm{Hom}_{\textbf{Graphs}}(X, Y)] \xrightarrow{\cong} \mathrm{Hom}_{\mathbb{Z}}(GX, GY)$$

THE END

Rigid systems of groups

Theorem (Shelah, 1974)

For any infinite cardinal κ there exists a system of groups $\{M_i\}_{i<2^\kappa}$ such that $|M_i| = \kappa$ and if $h : M_i \to M_j$ is a nonzero homomorphism then $i = j$ and h is a multiplication by an integer.

Alternative proof.

Vopěnka and Hedrlín (1965) proved that for any infinite cardinal κ there exists a rigid system of graphs $\{X_i\}_{i<2^\kappa}$, each graph of cardinality κ. Let $M_i = G_{fin}X_i$.

Generalized pure subgroups

If κ is an infinite cardinal, a subgroup N of M is said to be κ-pure if N is a direct summand of every subgroup N' such that $N \subseteq N' \subseteq M$ and $|N'/N| < \kappa$.

Theorem (Megibben, 1972)

For every infinite cardinal κ there exists a group M containing a κ-pure subgroup N which is not κ^+-pure.

Alternative proof.

Let X_α be the graph representing the order relation of the ordinal α. Let $N = G_{fin}X_\kappa$ and $M = G_{fin}(X_\kappa \vee X_{\kappa+1})$.

A class of groups

Theorem

There exists a class of groups M_α indexed by all ordinals α such that for $\alpha < \beta$ we have $\mathrm{Hom}(M_\beta, M_\alpha) = 0$.

Proof.

Let X_α be the graph representing the order relation of α.
Let $M_\alpha = GX_\alpha$.

The following are equivalent:

1. Negation of Vopěnka's Principle
2. There exists a rigid class of graphs
3. There exists a rigid class of groups

$1 \iff 2$ Definition
$3 \implies 2$ Adámek, Rosický, Trnková 1988

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Chains of group localizations

For any ordinal λ there exists a chain of groups M_α, $\alpha < \lambda$ such that the inclusions $M_\alpha \subseteq M_\beta$ are localizations for $\alpha < \beta < \lambda$. That means they induce isomorphisms

$$\mathrm{Hom}(M_\beta, M_\beta) \cong \mathrm{Hom}(M_\alpha, M_\beta)$$

The following are equivalent:

1. Negation of Vopěnka's Principle
2. $\mathcal{O}rd$ fully embeds into $\mathcal{G}raphs$
3. The chain above may be indexed by all ordinals

$3 \implies 2 \iff 1$ Adámek, Rosický, Trnková 1988

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Properties of G : **Graphs** $\longrightarrow$ **AbelianGroups**

1. $\mathbb{Z}[\mathrm{Hom}_{\mathbf{Graphs}}(X, Y)] \xrightarrow{\cong} \mathrm{Hom}_{\mathbb{Z}}(GX, GY)$
2. $f \perp Z$ if and only if $Gf \perp GZ$
3. the cardinality of GX is at least the continuum
4. G preserves countably directed colimits
5. G preserves monomorphisms
6. for every $a \in GX$ there exists a countable subgraph $C \subseteq X$ such that $a \in GC$
7. G does not preserve products.