# Milnor-Thurston homology of some wild topological spaces

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# Milnor-Thurston homology

Let $X$ be a topological space

$\mathcal{C}_k(X)$ consists of measures $\mu$ on $C(\Delta^k, X)$ with the following properties

- $\mu$ is a signed Borel measure
- $\mu$ is finite on any Borel set
- $\mu$ has a compact carrier

The boundary operator is defined like in singular theory:

**The boundary operator**

$$\partial = \sum_{i=0}^{k} (-1)^i \partial_i,$$

where $\partial_i : C(\Delta^k, X) \to C(\Delta^{k-1}, X)$ is induced by the inclusion of $\Delta^{k-1}$ as a face of $\Delta^k$

Homology is defined in the usual way:

**Homology**

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# Properties of Milnor-Thurston homology

- Milnor-Thurston homology satisfies Eilenberg-Steenrod Axioms
- As a consequence it coincides with singular homology for CW-complexes and...
- ... the Mayer-Vietoris theorem is true in this theory

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# Problems

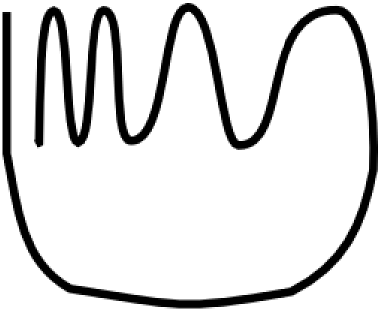
Our general goal is to investigate the behaviour of Milnor-Thurston homology theory for non-tame topological spaces

The first example we got interested in is the Warsaw Circle

# The Warsaw Circle

Let $W$ denote The Warsaw Circle

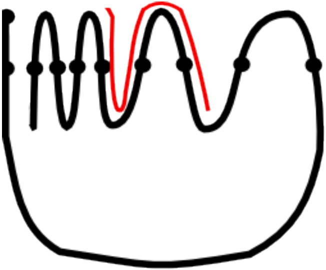
- Is $\mathcal{H}_1(W) = \mathbb{R}$?
- Is $\mathcal{H}_0(W) = \mathbb{R}$?
- What are the higher homology groups?

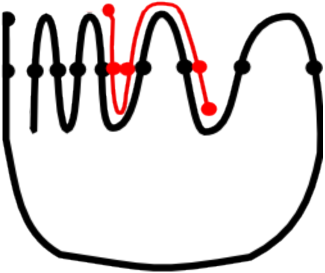


# Dividing singular simplices

**Question**

How can we formally achieve simplices division?

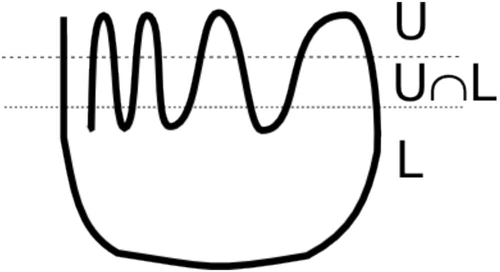




# Using Mayer-Vietoris theorem

## Answer

Formal solution of the problem is to use the Mayer-Vietoris theorem



# Using Mayer-Vietoris theorem (cont.)

The Mayer-Vietoris sequence reduces to

$$0 \to \mathcal{H}_1(W) \to \mathcal{H}_0(U \cap L) \xrightarrow{\ f\ } \mathcal{H}_0(U) \oplus \mathcal{H}_0(L) \to \mathcal{H}_0(W) \to 0.$$

From that

**Answer**

$\mathcal{H}_1(W)$ is the kernel of $f$.

We can see that $\mathcal{H}_0(U \cap L) \cong \mathcal{H}_0(U) \cong \mathcal{H}_0(L) \cong \ell^1$. Hence

**Observation**

$$f : \ell^1 \to \ell^1$$

# Using Mayer-Vietoris theorem (cont.)

The function $f : \ell^1 \to \ell^1, (m_k) \mapsto (x_k)$ of the previous frame is defined with the formulas

$$x_0 = m_0$$

$$x_k = m_k + m_{k-1}$$

The kernel of $f$ is trivial. Hence

**Solution**

$$\mathcal{H}_1(W) \cong 0$$

Additionally

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$$\mathcal{H}_0(W) \cong \ell^1 / f(\ell^1)$$

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# Berlanga Topology

Let $f$ be a real function on $C(\Delta^k, X)$. We define linear functional $\Lambda_f : \mathcal{C}_k(X) \to \mathbb{R}$ with the formula

$$\Lambda_f : \mu \mapsto \int_{C(\Delta^k, X)} f d\mu$$

We consider the weakest topology on $\mathcal{C}_k(X)$ such that each $\Lambda_f$ is continuous.

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# The zeroth homology group of the Warsaw Circle

We have seen that

$$\mathcal{H}_0(W) \cong \ell^1 / f(\ell^1)$$

Thus every homology class can be described by a sequence $x_k$. A simple calculation shows that taking

$$x_k = (-1)^k \left( \frac{1}{(k+2)^\beta} - \frac{1}{(k+1)^\beta} \right), \quad \text{for } 0 < \beta < 1$$

We get continuum independent homology classes. Hence

**Solution**

$$\mathcal{H}_0(W) \cong \mathbb{R}^{\mathfrak{c}}$$

Moreover. Taking similar sequences we prove that $\mathcal{B}_0(W)$ is not closed in $\mathcal{Z}_0(W)$. Thus

**Answer to Berlanga's question**

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# What can we prove further?

- Is $H_k(X) \to \mathcal{H}_k(X)$ always an injection?
- Are there some natural conditions when $\mathcal{H}_0(X)$ coincides with $H_0(X)$?

# Injectivity

**Theorem**

If $X$ has Borel path components, then $H_k(X) \to \mathcal{H}_k(X)$ is an injection

In order to prove this theorem it is enough to show that there does not exist a measure $\nu \in \mathcal{C}_1(X)$ such that $\partial\nu = \delta_{x_0} - \delta_{x_1}$ if $x_0, x_1 \in X$ lie in a different path components.

# Peano Continua

**Definition**

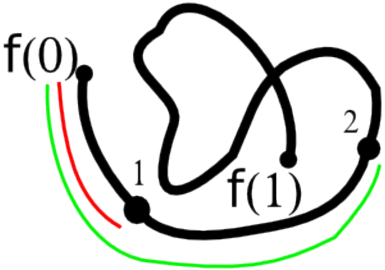
Peano continuum is a continuum that is locally connected

**Theorem**

If $X$ is a Peano continuum, then $\mathcal{H}_0(X)$ coincides with $H_0(X)$

# Sketch of the proof

- There is $f : [0,1] \to X$ that is onto
- Let $\mu$ is a measure on X - that is $\mu$ is a 0-chain
- We look for a measure $\nu$ on $C(\Delta^1, X)$ such that $\partial\nu = \mu - \mu(X)\delta_{f(0)}$



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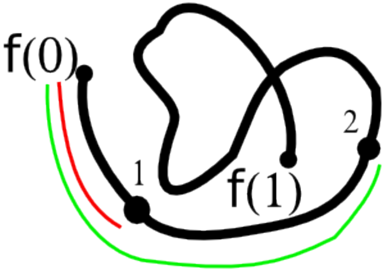

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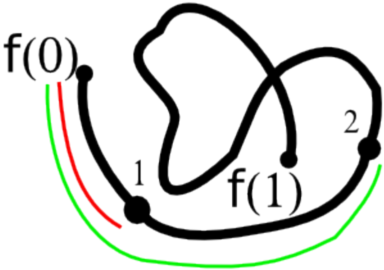

# Sketch of the proof (cont.)

In fact the following Lemma is the key to the proof

**Theorem**

If $f : [0,1] \to X$ is onto, and $\mu$ is measure on $X$ then there exists a measure $\tilde{\mu}$ on $[0,1]$ such that $f\tilde{\mu} = \mu$.

Then we use the natural map $g : [0,1] \to C(\Delta^1, X)$

$$\nu = g\tilde{\mu}$$

The measure $\nu$ has desired properties!

# Interesting conjectures

- Is $H_0(X) \to \mathcal{H}_0(X)$ is an injection when $X$ is a metric space?
- Under what condition $H_k(X) \to \mathcal{H}_k(X)$ is an injection for $k > 0$?
- Does $H_k(X)$ and $\mathcal{H}_k(X)$ coincide when $X$ is locally contractible?