Loop spaces and homological algebra

Warsaw, July 23, 2013

Haynes Miller

My theme:

"During the last decade the methods of algebraic topology have invaded extensively the domain of pure algebra, and initiated a number of internal revolutions. The purpose of this book is to present a unified account of these developments and to lay the foundations of a full-fledged theory."

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– H. Cartan, S. Eilenberg, September, 1953

Preface to **Homological Algebra**.

Question: How are $H_*(X)$ and $H_*(\Omega^n X)$ related? (coefficients in $\mathbb{F}_2$)

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I. The stable case: X a spectrum; how are $H_*(X)$ and $H_*(\Omega^\infty X)$ related?

There's a map:

$$\overline{H}_*(\Omega^\infty X) \to H_*(X)$$

arising from

$$\Sigma^\infty \Omega^\infty X \to X$$

which is part of the adjoint pair

$$\Sigma^\infty : \mathbf{Top}_* \rightleftarrows \mathbf{Sp} : \Omega^\infty$$

First answer (HM, 1975)

Structure on $H_*(\Omega^\infty X)$:
– Pontryagin product
– Kudo-Araki/Dyer-Lashof operations (R).

All killed by our map, so

$$\begin{array}{ccc} \overline{H}_*(\Omega^\infty X) & \longrightarrow & H_*(X) \\ \downarrow & & \uparrow h \\ QH_*(\Omega^\infty X) & \longrightarrow & \mathbb{F}_2 \otimes_R QH_*(\Omega^\infty X) \end{array}$$

(1) The natural transformation h is an iso if $X = \Sigma^\infty A$.

(2) Any spectrum can be resolved into suspension spectra.

So there's a spectral sequence

$$L_*(\mathbb{F}_2 \otimes_R Q)(H_*(\Omega^\infty X)) \Longrightarrow H_*(X)$$

“Resolution”: relative homological algebra (ref: Sammy Eilenberg and John Moore)

“surjection”: a map $Y \to X$ such that $\Omega^\infty Y \to \Omega^\infty X$ has a section.

“projective”: a retract P of a suspension spectrum.

$H_*(\Omega^\infty P)$ is then projective in

aRHa = allowable R-Hopf algebras.

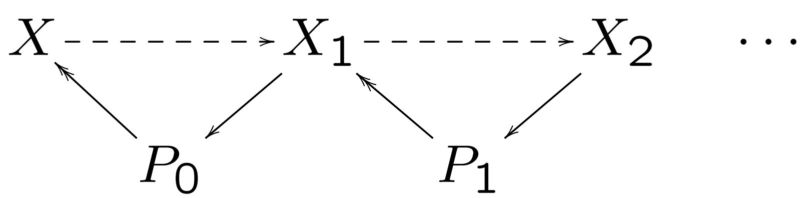
“Resolution”:

There are “enough projectives”: e.g.

$$X \leftarrow \Sigma^\infty \Omega^\infty X \tag{1}$$

is a surjection from a projective.

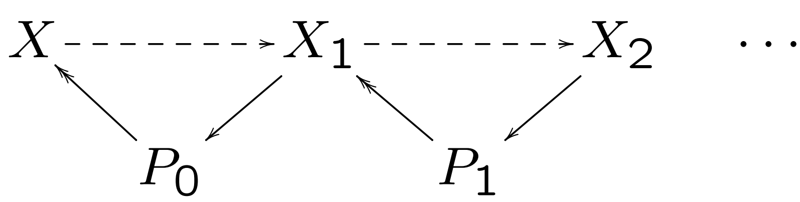
Taking fibers, inductively get a diagram



in which the bottom row is a projective resolution.

(1) gives the canonical resolution, but there are other interesting resolutions (e.g. the “Whitehead conjecture” gives an example for $X = H\mathbb{Z}$).

“Resolution”:



Apply homology to get an exact couple and so spectral sequence with

$$E_s^1 = H_*(P_s) = \mathbb{F}_2 \otimes_R QH_*(\Omega^\infty P_s)$$

since the map h is iso for projectives.
On the other hand,

$$H_*(\Omega^\infty X) \leftarrow H_*(\Omega^\infty P_0) \leftarrow H_*(\Omega^\infty P_1) \leftarrow \cdots$$

is a projective resolution of $H_*(\Omega^\infty X)$ as an allowable Hopf algebra over the Dyer-Lashof algebra, so

$$E_s^2 = L_s(\mathbb{F}_2 \otimes_R Q)(H_*(\Omega^\infty X))$$

Calculation of

$$E^2_s = L_s(\mathbb{F}_2 \otimes_R Q)(H_*(\Omega^\infty X))$$

$$(\textbf{aRHa}) \xrightarrow{Q} \textbf{1-aR-mod} \xrightarrow{\mathbb{F}_2 \otimes_R -} \text{g}\textbf{vs}$$

All these are abelian categories, the functors are additive, and Q takes projectives to projectives. So there is a Grothendieck spectral sequence (for $H \in$ **aRHa**)

$$\text{Untor}^R_*(\mathbb{F}_2, L_*Q(H))) \Longrightarrow L_*(\mathbb{F}_2 \otimes_R Q)(H)$$

Now $\qquad L_0Q(H) = QH = \text{Tor}^H_1(\mathbb{F}_2, \mathbb{F}_2),$

$$L_1Q(H) = Q\text{Tor}^H_2(\mathbb{F}_2, \mathbb{F}_2)$$

and there are no higher derived functors. The middle category admits a theory of Koszul resolutions (following work of Stewart Priddy and of Pete Bousfield and Ed Curtis).

Calculation proves collapse at E^2 if $X = H\mathbb{Z}$.

Other cases have been computed as well. For example David Kraines and Tom Lada worked out the case $X = \Sigma\mathrm{ko}$.

The spectrum associated to the infinite loop space $GL_1(S)$ of stable self equivalences of spheres would be very interesting to understand this way.

Second answer
(joint work with Rune Haugseng, 2012)

Structure on $H^*(\Omega^\infty X)$:
– unstable module over Steenrod algebra $\mathcal{A}$
– cup product

$$\mathcal{U} : \text{unstable } \mathcal{A}\text{-modules}, \ \mathsf{Sq}^i x = 0 \text{ for } i > |x|$$

$$\mathcal{K} : \text{unstable } \mathcal{A}\text{-algebras}, \ \mathsf{Sq}^n x = x^2, \text{ for } n = |x|$$

$$DM = M/(\mathsf{Sq}^i x : i > |x|)$$

$$UM = \mathsf{Sym}(M)/(\mathsf{Sq}^n x = x^2 : n = |x|)$$

$$\begin{array}{ccc} H^*(X) & \longrightarrow & H^*(\Omega^\infty X) \\ \downarrow & & \uparrow h \\ DH^*(X) & \longrightarrow & UDH^*(X) \end{array}$$

$$\mathcal{A}\text{-}\mathbf{mod} \xrightarrow{D} \mathcal{U} \xrightarrow{U} \mathcal{K}$$

$$\begin{array}{ccc} H^*(X) & \longrightarrow & H^*(\Omega^\infty X) \\ \downarrow & & \uparrow h \\ DH^*(X) & \longrightarrow & UDH^*(X) \end{array}$$

(1) h is an iso if X is a mod 2 GEM

(2) Any spectrum can be resolved into mod 2 GEMs

So there's a spectral sequence with

$$E_2^{-s} = L_s(UD)(H^*(X)) \Longrightarrow H^*(\Omega^\infty X)$$

In more detail . . .

"Resolution into mod 2 GEMs" via the cosimplicial spectrum

$$H \wedge X \Longrightarrow H^{\wedge 2} \wedge X \Rrightarrow H^{\wedge 3} \wedge X \cdots$$

Apply Ω^∞ and study the cohomology spectral sequence for the resulting cosimplicial space (Bousfield). Since

$$h : UDH^*(X) \longrightarrow H^*(\Omega^\infty X)$$

is iso on mod 2 GEMs,

$$H^*(\Omega^\infty(H^{\wedge(s+1)} \wedge X)) \xrightarrow{\cong} UD(H^*(H^{\wedge(s+1)} \wedge X))$$

$$\xrightarrow{\cong} UD(\mathcal{A}^{\otimes(s+1)} \otimes H^*(X))$$

so

$$E_2^{-s} = L_s(UD)(H^*(X))$$

Calculation of $L_*(UD)$:

Given a simplicial resolution

$$M \leftarrow P_\bullet \quad \text{in} \quad \mathsf{s}(\mathcal{A}\text{-}\mathbf{mod})$$

we want to calculate

$$L_*(UD)(M) = \pi_*(UDP_\bullet)$$

Two steps:

(1) Compute $\pi_*(DP_\bullet) = L_*D(M) \in \mathsf{g}\mathcal{U}$

(2) Given $V_\bullet \in \mathsf{s}\mathcal{U}$, compute $\pi_*(UV_\bullet)$ from $\pi_*(V_\bullet)$.

First Step: Compute $\pi_*(DP_\bullet) = L_*D(M)$

Warning: If $M \in \mathcal{U}$

$$\mathrm{Ext}_{\mathcal{U}}^{s,t}(M, \mathbb{F}_2)^\vee = L_s D(\Sigma^{-t} M)^0$$

so you can't hope for an explicit calculation if $t \geq s$. But:

Bill Singer; Jean Lannes and Said Zarati; Geoffrey Powell; others:–

$$(\Phi M)^i = \begin{cases} M^{i/2} & i \text{ even} \\ 0 & i \text{ odd} \end{cases}$$

Theorem. (Lannes and Zarati, after Singer) There are functors $R_s : \mathcal{U} \to \mathcal{U}$ such that

$$R_s(\mathbb{F}_2) = \mathbb{F}_2[t_1, \ldots, t_s]^{GL_s} = \mathbb{F}_2[c_{s,0}, \ldots, c_{s,s-1}]$$

and for $M \in \mathcal{U}$ and $s > t$

(1) $L_s D(\Sigma^{-t} M) = \Sigma R_s(\Sigma^{s-t-1} M)$

(2) In g**vs**, $R_s(M) = \Phi^s M \otimes R_s(\mathbb{F}_2)$

Second Step: Given $V_\bullet \in s\mathcal{U}$, determine $\pi_*(UV_\bullet)$ in terms of $\pi_*(V_\bullet)$.

Dold's theorem: Let $F : \mathbf{vs} \to \mathbf{vs}$. There is a functor $\mathcal{F} : g\mathbf{vs} \to g\mathbf{vs}$ and natural isomorphism $\pi_*(FV_\bullet) = \mathcal{F}(\pi_*(V_\bullet))$.

. . . doesn't apply; but not too far off:

$$\begin{array}{ccccc} \mathcal{U} & \xrightarrow{U} & \mathcal{K} & & \\ \downarrow & & \downarrow & & \\ \mathbf{rvs} & \xrightarrow{U} & g\mathbf{ca} & \longrightarrow & g\mathbf{vs} \end{array}$$

rvs = restricted vector spaces: graded V with $\phi : V^i \to V^{2i}$, such that $\phi | V^0 = 1$.

g**ca** = graded commutative algebras

$UV = \mathrm{Sym}(V)/x^2 = \phi x$

rvs is like the category of modules over a PID; any object breaks up as a direct sum of $\mathbb{F}_2$'s, $F[q]$'s, and $F[q,k]$'s, where

$$F[q] = \langle \iota_q, \phi\iota_q, \ldots \rangle$$

and $F[q,k]$ is defined by the short exact sequence

$$0 \longleftarrow F[q,k] \longleftarrow F[q] \overset{\phi^k}{\longleftarrow} F[2^k q] \longleftarrow 0$$

Key objects in s**rvs**: $K[n,q]$ and $K[n,q,k]$ such that

$$\pi_n(K[n,q]) = F[q] \quad , \quad \pi_n(K[n,q,k]) = F[q,k]$$

and zero otherwise. $UF[q] = \mathrm{Sym}[q]$, so

$$UK[n,q] = \mathrm{Sym}[n,q]$$

The homotopy of this simplicial commutative algebra was worked out by Cartan (using earlier ideas of Eilenberg and Mac Lane), and later by Bousfield and by Bill Dwyer.

We have a cofiber sequence in sg**ca**

$$K[n,q] \to K[n,q,k] \to K[n+1, 2^k q]$$

This is like the fiber sequence in $\mathbf{Top}_*$

$$K(\mathbb{Z},n) \to K(\mathbb{Z}/2^k\mathbb{Z},n) \to K(\mathbb{Z},n+1)$$

The Serre spectral sequence for this collapses, showing that (over $\mathcal{A}$ if $k > 1$)

$$H^*(K(\mathbb{Z}/2^k,n)) \cong H^*(K(\mathbb{Z},n)) \otimes H^*(K(\mathbb{Z},n+1))$$

The same strategy computes the homotopy of $UK[n,q,k]$, and gives a *non-functorial* formula for $\pi_*(UV_\bullet)$ in terms of $\pi_*(V_\bullet)$.

One outcome:

Theorem (RH and HM). This spectral sequence collapses if X is a suspension spectrum.

Our proof at this point is a counting argument; we know what $H^*(\Omega^\infty\Sigma^\infty A)$ is, and compute that the E_2 term has the same Poincaré series.

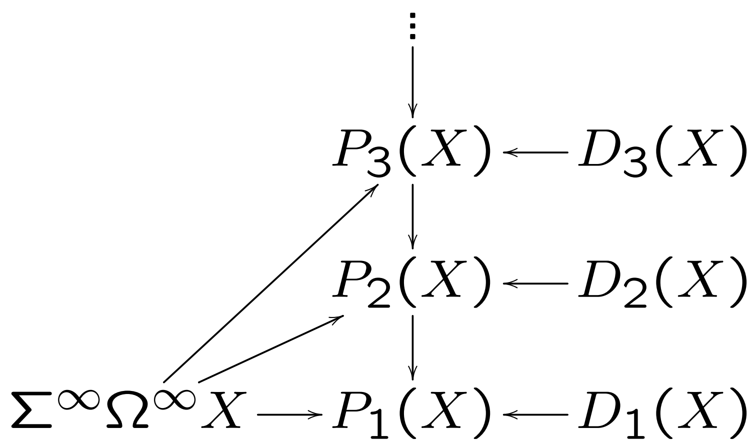
There is also a . . .

Third answer

(Nick Kuhn and Justin McCarty)

Consider the Taylor tower for the functor

$$\Sigma^\infty\Omega^\infty : \mathbf{Sp} \to \mathbf{Sp}$$



$$D_s(X) = (X^{\wedge s})_{h\Sigma_s}$$

The associated homology spectral sequence has

$$E^1_{s,t} = H_{s+t}(D_s(X)) \Longrightarrow \overline{H}_{s+t}(\Omega^\infty X)$$

This E_1 term is the answer in case X is a suspension spectrum: in that case this spectral sequence collapses automatically.

Kuhn and McCarty conduct an interesting investigation into this spectral sequence.

II. An analogy

How is $\pi_*(X)$ is related to $H_*(X)$?

Hurewicz map $\pi_*(X) \to H_*(X)$

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Hurewicz map $\pi_*(X) \to H_*(X)$

First Answer (Bousfield and Kan) Hurewicz factors as

$$\begin{array}{ccc} \pi_*(X) & \longrightarrow & H_*(X) \\ h \downarrow & & \uparrow \\ \mathbb{F}_2 \square_{\mathcal{A}_*} PH_*(X) & \longrightarrow & PH_*(X) \end{array}$$

(1) h is an iso if X is a mod 2 GEM

(2) Any space can be resolved into mod 2 Eilenberg Mac Lane spaces (via the Bousfield-Kan cosimplicial space)

The unstable Adams spectral sequence results:

$$R^*(\mathbb{F}_2 \square_{\mathcal{A}_*} P)(H_*(X)) \Longrightarrow \pi_*(X)$$

Second answer (Chris Stover and David Blanc) Hurewicz factors as

$$\begin{array}{ccc}
\pi_*(X) & \longrightarrow & \overline{H}_*(X) \\
\downarrow & & \uparrow h \\
\dfrac{\pi_*(X)}{\text{compositions}} & \longrightarrow & \dfrac{\pi_*(X)}{\text{compositions and } [-,-]} \\
 & & \| \\
 & & Q\pi_*(X)
\end{array}$$

(1) h is a natural transformation which is an isomorphism if X is a wedge of spheres

(2) Any space can be resolved into wedges of spheres (via the Stover resolution)

Get the "Hurewicz spectral sequence"

$$L_*Q(\pi_*(X)) \Longrightarrow \overline{H}_*(X)$$

There is also a . . .

Third answer: (Bousfield, Curtis, Kan, Quillen, Rector, Schlesinger)

The mod 2 lower central series filtration on the Kan loop group of a reduced simplicial set yields a spectral sequence converging to $\pi_*(X)$.

Bousfield and Curtis: In certain cases it is closely related to the Bousfield-Kan Adams spectral sequence, but it's definitely distinct.

III. Finite loops

How are $H_*(X)$ and $H_*(\Omega^n X)$ related to each other?

$$\overline{H}_*(\Omega^n X) \to \Sigma^{-n}\overline{H}_*(X)$$

from the adjoint pair

$$\Sigma^n : \mathbf{Top}_* \rightleftarrows \mathbf{Top}_* : \Omega^n$$

First answer

n-fold loop spaces have operations (Fred Cohen), which die under this map:

$$\begin{array}{ccc} \overline{H}_*(\Omega^n X) & \longrightarrow & \Sigma^{-n}\overline{H}_*(X) \\ & \searrow \quad \nearrow_{h} & \\ & Q_n H_*(\Omega^n X) & \end{array}$$

$$\begin{array}{ccc} \overline{H}_*(\Omega^n X) & \longrightarrow & \Sigma^{-n}\overline{H}_*(X) \\ & \searrow \quad \nearrow_{h} & \\ & Q_n H_*(\Omega^n X) & \end{array}$$

(1) h is an isomorphism if $X = \Sigma^n A$.

(2) Any space resolves into n-fold suspensions via

$$X \leftarrow \Sigma^n \Omega^n X \Leftarrow \Sigma^n \Omega^n \Sigma^n \Omega^n X \Lleftarrow \cdots$$

Resulting spectral sequence:

$$L_* Q_n(H_*(\Omega^n X)) \Rightarrow \Sigma^{-n}\overline{H}_*(X)$$

studied in some cases by Birgit Richter and Stephanie Ziegenhagen via a composite functor spectral sequence (Blanc, Stover).

Example: $\mathbf{n = 1}$.

The universal structure on the homology of a single loop space is that of an associative algebra.

$$L_*Q(H_*(\Omega X)) \Rightarrow \Sigma^{-1}\overline{H}_*(X)$$

Quillen proved that

$$L_*Q(A) = \mathrm{Tor}_{*+1}^{A \otimes A^{\mathrm{op}}}(\mathbb{F}_2, A)$$

and if A is a Hopf algebra then this gives

$$L_*Q(A) = \mathrm{Tor}_{*+1}^{A}(\mathbb{F}_2, \mathbb{F}_2)$$

So this spectral sequence looks like a reduced form of the Rothenberg-Steenrod spectral sequence.

Question: (Richter) Is it?

Second answer

$$\begin{array}{ccc} \Sigma^{-n}\overline{H}^*(X) & \longrightarrow & H^*(\Omega^n X) \\ \downarrow & & \uparrow h \\ \Omega^n Q H^*(X) & \longrightarrow & U\Omega^n Q H^*(X) \end{array}$$

Notation: $\Omega^n = D\Sigma^{-n} : \mathcal{U} \to \mathcal{U}$

(1) h is an iso for X a mod 2 GEM.

(2) Any space admits a resolution into mod 2 GEMs (Bousfield-Kan)

So we get a spectral sequence

$$E_2^{-s} = L_s(U\Omega^n Q)(H^*(X)) \Longrightarrow H^*(\Omega^n X)$$

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The hard part here is L_*Q, André-Quillen homology.

But suppose $H^*(X)$ is polynomial (e.g. various compact Lie groups). Then

$$L_i Q(H^*(X)) = 0 \text{ for } i > 0\,.$$

When $n = 1$, one finds that

$$H^*(\Omega X) = U(\Sigma^{-1} Q H^*(X))$$

An alternative simplification is . . .

Very nice case: $H^*(X) = U(M)$

If $P_\bullet$ is a simplicial resolution of UM, then

$$\pi_*(\Omega^n QUP_\bullet) = L_*\Omega^n(M)$$

so we get a very explicit E_2 term for the spectral sequence.

For example, if $n = 1$,

$$E_2^* = U\Omega M \otimes \Gamma[sL_1\Omega(M)]$$

Very nice case: $H^*(X) = U(M)$

John Harper and HM, 1978: Loop down the Massey-Peterson tower.

Theorem. There is a Hopf algebra filtration of $H^*(\Omega^n X)$ such that

$$A_{-1} = \mathbb{F}_2 \quad , \quad \bigcup A_s = H^*(\Omega^n X)$$

with Hopf quotients

$$A_{s+1}//A_s = U(\Omega L_{s+1}\Omega^{s+n}(M))$$

. . . *provided* that X is also simply connected, of finite type, and such that if

$$b = \min\{i : \overline{H}^i(X) \neq 0\}$$

then

$$H^i(X) = 0 \text{ for } i > 4(b - n + 1)\,.$$

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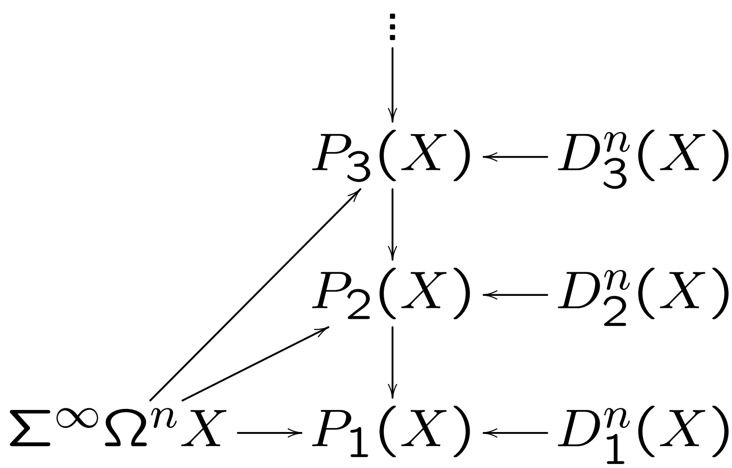
The Harper-Miller answer is our E_2, and the range condition precisely guarantees that the spectral sequence collapses.

There is also a

Third answer (Kuhn; Büscher, Hebestreit, Röndigs, Stelzer)

The Taylor tower for the functor

$$\Sigma^\infty \Omega^n : \mathbf{Top}_* \to \mathbf{Sp}$$



$$D_s^n(X) = \Sigma^{-n} C(\mathbb{R}^n, s)_+ \wedge_{\Sigma_s} (\Sigma^\infty X)^{\wedge s}$$

This gives a spectral sequence

$$E^1_{s,t} = H_{s+t}(D_s^n(X)) \Longrightarrow \overline{H}_{s+t}(\Omega^n X)$$

Hey, for relating $H_*(X)$ and $H_*(\Omega X)$ we also have the Eilenberg-Moore spectral sequence!

Question: How does that fit into this schema?

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Question: How does that fit into this schema?

Answer: Left to the reader!