Power Series in Synthetic Differential Geometry

Davorin Lešnik

Department of Mathematics
Darmstadt University of Technology, Germany
lesnik@mathematik.tu-darmstadt.de

Warszawa, July 26th, 2013

Synthetic differential geometry

SDG is a setting, where analytical notions (such as the derivative) are given algebraically/axiomatically, rather than via standard limit arguments.

- Achieved by allowing (nontrivial) **infinitesimals**. In particular $D := \{x \in \mathbf{R} \mid x^2 = 0\} \neq \{0\}$, where $\mathbf{R}$ are the **smooth reals**.
- Necessitates constructive logic. Models of SDG are certain topoi.
- Limit arguments impossible ($\mathbf{R}$ not Hausdorff). In particular $\{0\} \subsetneq \mathbf{R}_{[0,0]} \subseteq \bigcap_{n \in \mathbb{N}_{>0}} \mathbf{R}_{(-1/n,1/n)}$.
- But: Closer to the usual intuition (especially physics).
- Definitions and proofs can simplify significantly. E.g. the tangent bundle of X is given as X^D; the tangent bundle projection is ev_0.
- Theorems in SDG imply analogous theorems in classical analysis (usually in more general version).

Some axioms in SDG

Kock-Lawvere axiom: For every map $f : D \to \mathbf{R}$ there exist unique $a, b \in \mathbf{R}$ such that $f(x) = a \cdot x + b$ for all $x \in D$.

- Allows defining derivatives. In particular, every real-valued map, defined on any interval, has a derivative ("all maps are smooth").

Some axioms in SDG

Kock-Lawvere axiom: For every map $f : D \to \mathbf{R}$ there exist unique $a, b \in \mathbf{R}$ such that $f(x) = a \cdot x + b$ for all $x \in D$.

- Allows defining derivatives. In particular, every real-valued map, defined on any interval, has a derivative ("all maps are smooth").

Constancy principle: For any interval $I \subseteq \mathbf{R}$ and $f : I \to \mathbf{R}$, if the derivative f' is constantly 0, then f is constant.

Some axioms in SDG

Kock-Lawvere axiom: For every map $f\colon D \to \mathbf{R}$ there exist unique $a, b \in \mathbf{R}$ such that $f(x) = a \cdot x + b$ for all $x \in D$.

- Allows defining derivatives. In particular, every real-valued map, defined on any interval, has a derivative ("all maps are smooth").

Constancy principle: For any interval $I \subseteq \mathbf{R}$ and $f\colon I \to \mathbf{R}$, if the derivative f' is constantly 0, then f is constant.

Integration axiom: For every map $f\colon \mathbf{R}_{[0,1]} \to \mathbf{R}$ there exists a unique map $g\colon \mathbf{R}_{[0,1]} \to \mathbf{R}$ such that $g' = f$ and $g(0) = 0$.

- Allows integration. In particular, every real-valued map, defined on any interval, has a primitive function ("all maps are integrable").
- Implies the Constancy principle.

Some axioms in SDG

Kock-Lawvere axiom: For every map $f\colon D \to \mathbf{R}$ there exist unique $a, b \in \mathbf{R}$ such that $f(x) = a \cdot x + b$ for all $x \in D$.

- Allows defining derivatives. In particular, every real-valued map, defined on any interval, has a derivative ("all maps are smooth").

Constancy principle: For any interval $I \subseteq \mathbf{R}$ and $f\colon I \to \mathbf{R}$, if the derivative f' is constantly 0, then f is constant.

Integration axiom: For every map $f\colon \mathbf{R}_{[0,1]} \to \mathbf{R}$ there exists a unique map $g\colon \mathbf{R}_{[0,1]} \to \mathbf{R}$ such that $g' = f$ and $g(0) = 0$.

- Allows integration. In particular, every real-valued map, defined on any interval, has a primitive function ("all maps are integrable").
- Implies the Constancy principle.

Existence of certain functions: There exist unique maps $\exp, \cos, \sin\colon \mathbf{R} \to \mathbf{R}$ which solve the following IVPs respectively.

$$\exp' = \exp \qquad \cos'' + \cos = 0 \qquad \sin'' + \sin = 0$$
$$\exp(0) = 1 \qquad \cos(0) = 1,\ \cos'(0) = 0 \qquad \sin(0) = 0,\ \sin'(0) = 1$$

Talk

Purpose: We find the axiom allowing us to define **power series** in SDG and observe some of its consequences.

Contents:

- **Preparation**
  We consider some classical properties of power series and make them suitable for constructive treatment.
- **Power series axiom**
  We state the axiom and see how it works.

Preparation

We work classically at first.

Informal definition: For $\rho \in \mathbb{R}_{>0}$ let A_ρ denote the set of all power series (with the center in 0) which converge on $\mathbb{R}_{(-\rho,\rho)}$.

Preparation

We work classically at first.

Informal definition: For $\rho \in \mathbb{R}_{>0}$ let A_ρ denote the set of all power series (with the center in 0) which converge on $\mathbb{R}_{(-\rho,\rho)}$.

For $\alpha \in \mathbb{R}^\mathbb{N}$ let r_α denote the radius of convergence of $\sum_{n\in\mathbb{N}} \alpha_n x^n$.

Hence $A_\rho = \{\alpha \in \mathbb{R}^\mathbb{N} \mid r_\alpha \geq \rho\}$.

Preparation

We work classically at first.

Informal definition: For $\rho \in \mathbb{R}_{>0}$ let A_ρ denote the set of all power series (with the center in 0) which converge on $\mathbb{R}_{(-\rho,\rho)}$.

For $\alpha \in \mathbb{R}^\mathbb{N}$ let r_α denote the radius of convergence of $\sum_{n\in\mathbb{N}} \alpha_n x^n$.

Hence $A_\rho = \{\alpha \in \mathbb{R}^\mathbb{N} \mid r_\alpha \geq \rho\}$.

Cauchy-Hadamard theorem: $r_\alpha = \dfrac{1}{\limsup_{n\to\infty} \sqrt[n]{|\alpha_n|}}$

Preparation

We work classically at first.

Informal definition: For $\rho \in \mathbb{R}_{>0}$ let A_ρ denote the set of all power series (with the center in 0) which converge on $\mathbb{R}_{(-\rho,\rho)}$.

For $\alpha \in \mathbb{R}^{\mathbb{N}}$ let r_α denote the radius of convergence of $\sum_{n\in\mathbb{N}} \alpha_n x^n$.

Hence $A_\rho = \{\alpha \in \mathbb{R}^{\mathbb{N}} \mid r_\alpha \geq \rho\}$.

Cauchy-Hadamard theorem: $r_\alpha = \dfrac{1}{\limsup_{n\to\infty} \sqrt[n]{|\alpha_n|}}$

Formal definition:
$A_\rho := \{\alpha \in \mathbb{R}^{\mathbb{N}} \mid \forall\, q \in \mathbb{R}_{(0,\rho)} \,.\, \exists\, m \in \mathbb{N} \,.\, \forall\, n \in \mathbb{N}_{\geq m} \,.\, (-1 < \alpha_n q^n < 1)\}$

Differential algebras

A_ρ is a

- unital commutative algebra (for convolution),
- equipped with a unary operation ("derivative") $\alpha'_n := (n+1) \cdot \alpha_{n+1}$ which is a linear map satisfying the **Leibniz rule** $(\alpha \cdot \beta)' = \alpha' \cdot \beta + \alpha \cdot \beta'$.

We say that A_ρ is a **differential algebra**.

Differential algebras

A_ρ is a

- unital commutative algebra (for convolution),
- equipped with a unary operation ("derivative") $\alpha'_n := (n+1) \cdot \alpha_{n+1}$ which is a linear map satisfying the **Leibniz rule** $(\alpha \cdot \beta)' = \alpha' \cdot \beta + \alpha \cdot \beta'$.

We say that A_ρ is a **differential algebra**.

A unital algebra morphism $f \colon A \to B$ is called a **diff. algebra morphism** when it additionally preserves derivation: $f(a') = f(a)'$ for all $a \in A$.

Differential algebras

A_ρ is a

- unital commutative algebra (for convolution),
- equipped with a unary operation ("derivative") $\alpha'_n := (n+1) \cdot \alpha_{n+1}$ which is a linear map satisfying the **Leibniz rule** $(\alpha \cdot \beta)' = \alpha' \cdot \beta + \alpha \cdot \beta'$.

We say that A_ρ is a **differential algebra**.

A unital algebra morphism $f \colon A \to B$ is called a **diff. algebra morphism** when it additionally preserves derivation: $f(a') = f(a)'$ for all $a \in A$.

Example: The map $\iota_\rho \colon A_\rho \to \mathcal{C}^\infty(\mathbb{R}_{(-\rho,\rho)})$, given by

$$\iota_\rho(\alpha) := x \mapsto \sum_{n \in \mathbb{N}} \alpha_n x^n,$$

is an (injective) differential algebra morphism.

Idea: Characterize ι_ρ as a certain differential algebra morphism.

A_ρ and $\mathcal{C}^\infty(\mathbb{R}_{(-\rho,\rho)})$ can be equipped with natural choices of topologies. It turns out that ι_ρ is continuous.

Theorem: For every $\rho \in \mathbb{R}_{>0}$ there exists a unique continuous differential algebra morphism $F\colon A_\rho \to \mathcal{C}^\infty(\mathbb{R}_{(-\rho,\rho)})$ with the property

$$F(\alpha)(v \cdot x) = F\big(n \mapsto \alpha_n v^n\big)(x)$$

for all $\alpha \in A_\rho$, $x \in \mathbb{R}_{(-\rho,\rho)}$ and $v \in \mathbb{R}_{[0,1]}$ (namely ι_ρ).

View $\mathbb{R}$ as a preorder category, with an arrow $\rho \to \sigma$ given for all $\rho, \sigma \in \mathbb{R}$, $\rho \geq \sigma$.

- $A_\square$ and $\mathcal{C}^\infty(\mathbb{R}_{(-\square,\square)})$ are functors from $\mathbb{R}$ to the category of differential algebras, where
- $\rho \to \sigma$ gets mapped
  - by $A_\square$ to the inclusion $A_\rho \hookrightarrow A_\sigma$, and
  - by $\mathcal{C}^\infty(\mathbb{R}_{(-\square,\square)})$ to the restriction mapping $\mathcal{C}^\infty(\mathbb{R}_{(-\rho,\rho)}) \to \mathcal{C}^\infty(\mathbb{R}_{(-\sigma,\sigma)})$, $f \mapsto f|_{\mathbb{R}_{(-\sigma,\sigma)}}$.
- Furthermore, ι is a natural transformation between these functors.

We get a "ladder with uncountably many rungs":

$$\begin{array}{ccccccc}
\dots & \longrightarrow & A_\rho & \longrightarrow & A_\sigma & \longrightarrow & \dots \\
 & & \downarrow{\scriptstyle \iota_\rho} & & \downarrow{\scriptstyle \iota_\sigma} & & \\
\dots & \longrightarrow & \mathcal{C}^\infty(\mathbb{R}_{(-\rho,\rho)}) & \longrightarrow & \mathcal{C}^\infty(\mathbb{R}_{(-\sigma,\sigma)}) & \longrightarrow & \dots
\end{array}$$

Observe that this ladder has a limit and a colimit:

$$\begin{array}{ccccccccc}
A_\infty & \longrightarrow & \dots & \longrightarrow & A_\rho & \longrightarrow & A_\sigma & \longrightarrow & \dots & \longrightarrow & A_0 \\
\downarrow{\scriptstyle \iota_\infty} & & & & \downarrow{\scriptstyle \iota_\rho} & & \downarrow{\scriptstyle \iota_\sigma} & & & & \downarrow{\scriptstyle \iota_0} \\
\mathcal{C}^\infty(\mathbb{R}) & \longrightarrow & \dots & \longrightarrow & \mathcal{C}^\infty(\mathbb{R}_{(-\rho,\rho)}) & \longrightarrow & \mathcal{C}^\infty(\mathbb{R}_{(-\sigma,\sigma)}) & \longrightarrow & \dots & \longrightarrow & \mathcal{C}^\infty_0(0)
\end{array}$$

- $\mathcal{C}^\infty_0(0)$ denotes the set of **germs** of smooth functions around 0.
- $A_\infty = \bigcap_{\rho \in \mathbb{R}_{>0}} A_\rho$ represents the set of **entire analytic functions**.
- $A_0 = \bigcup_{\rho \in \mathbb{R}_{>0}} A_\rho$ is the set of power series with positive radii of convergence.

Furthermore, for all $\rho, \sigma \in \mathbb{R}_{>0}$ define

$$\begin{aligned} &\phi_{\sigma\rho} \colon A_\rho \to A_\sigma && \psi_{\sigma\rho} \colon \mathcal{C}^\infty(\mathbb{R}_{(-\rho,\rho)}) \to \mathcal{C}^\infty(\mathbb{R}_{(-\sigma,\sigma)}) \\ &\phi_{\sigma\rho}(\alpha) := n \mapsto \alpha_n \cdot \left(\tfrac{\rho}{\sigma}\right)^n && \psi_{\sigma\rho}(f) := x \mapsto f\left(\tfrac{\rho}{\sigma} \cdot x\right) \end{aligned}$$

These are unital algebra morphisms, but not differential algebra morphisms.

They satisfy the **cocycle** conditions:

$$\phi_{\rho\rho} = \mathrm{Id}_{A_\rho}, \quad \phi_{\rho\sigma} = \phi_{\sigma\rho}^{-1}, \quad \phi_{\tau\sigma} \circ \phi_{\sigma\rho} = \phi_{\tau\rho},$$

and similarly for ψ.

This diagram commutes:

$$\begin{array}{ccc} A_\rho & \xrightarrow{\ \phi_{\sigma\rho}\ } & A_\sigma \\ {\scriptstyle \iota_\rho}\big\downarrow & & \big\downarrow{\scriptstyle \iota_\sigma} \\ \mathcal{C}^\infty(\mathbb{R}_{(-\rho,\rho)}) & \xrightarrow[\ \psi_{\sigma\rho}\ ]{} & \mathcal{C}^\infty(\mathbb{R}_{(-\sigma,\sigma)}) \end{array}$$

Hence $\iota_\rho = \phi_{\rho\sigma} \circ \iota_\sigma \circ \psi_{\sigma\rho}$.

Power series axiom

We now work in a synthetic setting (assuming of course the Kock-Lawvere axiom).
We additionally postulate the Constancy principle.

The definition
$A_\rho := \left\{\alpha \in \mathbf{R}^{\mathbb{N}} \;\middle|\; \forall\, q \in \mathbf{R}_{(0,\rho)} \,.\, \exists\, m \in \mathbb{N} \,.\, \forall\, n \in \mathbb{N}_{\geq m} \,.\, (-1 < \alpha_n q^n < 1)\right\}$
still makes sense.

Power series axiom

We now work in a synthetic setting (assuming of course the Kock-Lawvere axiom). We additionally postulate the Constancy principle.

The definition
$A_\rho := \left\{\alpha \in \mathbf{R}^{\mathbb{N}} \;\middle|\; \forall\, q \in \mathbf{R}_{(0,\rho)} \,.\, \exists\, m \in \mathbb{N} \,.\, \forall\, n \in \mathbb{N}_{\geq m} \,.\, (-1 < \alpha_n q^n < 1)\right\}$
still makes sense.

In view of the preceding preparation we propose the following axiom.

Axiom: There is a differential algebra morphism $\iota_1 \colon A_1 \to \mathbf{R}^{\mathbf{R}_{(-1,1)}}$, for which

$$\iota_1(\alpha)(v \cdot x) = \iota_1\big(n \mapsto \alpha_n v^n\big)(x)$$

for all $\alpha \in A_1$, $x \in \mathbf{R}_{(-1,1)}$ and $v \in \mathbf{R}_{[0,1]}$.

Some observation

- "Taylor polynomials": $\iota_1(\alpha)(x) = \left(\sum_{k\in\mathbb{N}_{<n}} \alpha_k x^k\right) + x^n \cdot \iota_1(S^n\alpha)(x)$ (where S is the shift map)
- $\big(\iota_1(\alpha)\big)^{(n)}(0) = n! \cdot \alpha_n \quad (\Rightarrow \iota_1$ is injective)
- all $x \in \mathbf{R}_{(0,\infty)}$ are invertible (use the trick with Neumann series)

We can thus define for $\rho, \sigma \in \mathbf{R}_{(0,\infty)}$:

$$\phi_{\sigma\rho} \colon A_\rho \to A_\sigma \qquad \psi_{\sigma\rho} \colon \mathbf{R}^{\mathbf{R}(-\rho,\rho)} \to \mathbf{R}^{\mathbf{R}(-\sigma,\sigma)}$$

$$\phi_{\sigma\rho}(\alpha) := n \mapsto \alpha_n \cdot \left(\tfrac{\rho}{\sigma}\right)^n \qquad \psi_{\sigma\rho}(f) := x \mapsto f\left(\tfrac{\rho}{\sigma} \cdot x\right)$$

Let $\iota_\rho \colon A_\rho \to \mathbf{R}^{\mathbf{R}(-\rho,\rho)}$ be given by $\iota_\rho := \psi_{\rho 1} \circ \iota_1 \circ \phi_{1\rho}$.

We again obtain the ladder:

$$\begin{array}{ccccccccc}
A_\infty & \longrightarrow & \dots & \longrightarrow & A_\rho & \longrightarrow & A_\sigma & \longrightarrow & \dots & \longrightarrow & A_0 \\
\downarrow{\scriptstyle \iota_\infty} & & & & \downarrow{\scriptstyle \iota_\rho} & & \downarrow{\scriptstyle \iota_\sigma} & & & & \downarrow{\scriptstyle \iota_0} \\
\mathbf{R}^{\mathbf{R}} & \longrightarrow & \dots & \longrightarrow & \mathbf{R}^{\mathbf{R}(-\rho,\rho)} & \longrightarrow & \mathbf{R}^{\mathbf{R}(-\sigma,\sigma)} & \longrightarrow & \dots & \longrightarrow & \mathcal{C}_0^\infty(0)
\end{array}$$

Examples

Note that $(n \mapsto \frac{1}{n!}) \in A_\rho$ for all $\rho \in \mathbf{R}_{(0,\infty)}$. Define $\boxed{\exp := \iota_\infty(n \mapsto \frac{1}{n!})}$.

Examples

Note that $(n \mapsto \frac{1}{n!}) \in A_\rho$ for all $\rho \in \mathbf{R}_{(0,\infty)}$. Define $\boxed{\exp := \iota_\infty(n \mapsto \frac{1}{n!})}$.

- We have $\iota_\infty(n \mapsto \frac{1}{n!})' = \iota_\infty\big((n \mapsto \frac{1}{n!})'\big) = \iota_\infty(n \mapsto \frac{1}{n!})$ and $\iota_\infty(n \mapsto \frac{1}{n!})(0) = \frac{1}{0!} = 1$. Thus exp solves the IVP $y' = y$, $y(0) = 1$.

Examples

Note that $(n \mapsto \frac{1}{n!}) \in A_\rho$ for all $\rho \in \mathbf{R}_{(0,\infty)}$. Define $\boxed{\exp := \iota_\infty(n \mapsto \frac{1}{n!})}$.

- We have $\iota_\infty(n \mapsto \frac{1}{n!})' = \iota_\infty((n \mapsto \frac{1}{n!})') = \iota_\infty(n \mapsto \frac{1}{n!})$ and $\iota_\infty(n \mapsto \frac{1}{n!})(0) = \frac{1}{0!} = 1$. Thus exp solves the IVP $y' = y$, $y(0) = 1$.
- Let $f, g \colon \mathbf{R} \to \mathbf{R}$ be any solutions of this IVP. Then

$$\big(f(x) \cdot g(-x)\big)' = f'(x) \cdot g(-x) - f(x) \cdot g'(-x) =$$

$$= f(x) \cdot g(-x) - f(x) \cdot g(-x) = 0$$

and $f(0) \cdot g(-0) = 1$, so by Constancy principle $f(x) \cdot g(-x) \equiv 1$. In particular (for $f = g$) we have $g(-x) = \frac{1}{g(x)}$, so $f(x) \equiv g(x)$.

Examples

Note that $(n \mapsto \frac{1}{n!}) \in A_\rho$ for all $\rho \in \mathbf{R}_{(0,\infty)}$. Define $\boxed{\exp := \iota_\infty(n \mapsto \frac{1}{n!})}$.

- We have $\iota_\infty(n \mapsto \frac{1}{n!})' = \iota_\infty\big((n \mapsto \frac{1}{n!})'\big) = \iota_\infty(n \mapsto \frac{1}{n!})$ and $\iota_\infty(n \mapsto \frac{1}{n!})(0) = \frac{1}{0!} = 1$. Thus exp solves the IVP $y' = y$, $y(0) = 1$.

- Let $f, g \colon \mathbf{R} \to \mathbf{R}$ be any solutions of this IVP. Then

$$\big(f(x) \cdot g(-x)\big)' = f'(x) \cdot g(-x) - f(x) \cdot g'(-x) =$$

$$= f(x) \cdot g(-x) - f(x) \cdot g(-x) = 0$$

  and $f(0) \cdot g(-0) = 1$, so by Constancy principle $f(x) \cdot g(-x) \equiv 1$. In particular (for $f = g$) we have $g(-x) = \frac{1}{g(x)}$, so $f(x) \equiv g(x)$.

In conclusion, exp is the unique solution to the above IVP. Similarly we get existence and uniqueness of cos and sin.

Power series of several variables

Denote by ${}^{+}\!\geq$ the following relation between tuples $t = (t_0, \dots, t_{L-1}) \in \mathbb{N}^L$ and natural numbers $m \in \mathbb{N}$:

$$t \ {}^{+}\!\geq m := \sum_{k \in \mathbb{N}_{<L}} t_k \geq m.$$

For $\rho = (\rho_0, \dots, \rho_{L-1}) \in \mathbf{R}^L_{(0,\infty)}$ generalize $A_\rho :=$

$$\left\{ \alpha \in \mathbf{R}^{\mathbb{N}^L} \;\middle|\; \forall\, q \in \prod_{k \in \mathbb{N}_{<L}} \mathbf{R}_{(0,\rho_k)} \,.\, \exists\, m \in \mathbb{N} \,.\, \forall\, t \in \mathbb{N}^L_{{}^{+}\!\geq m} \,.\, \left(-1 < \alpha_t \prod_{k \in \mathbb{N}_{<L}} q_k^{t_k} < 1\right) \right\}.$$

Inductively on L define $\iota_\rho \colon A_\rho \to \mathbf{R}^{\prod_{k \in \mathbb{N}_{<L}} \mathbf{R}_{(-\rho_k,\rho_k)}}$ as follows:

$$\iota_{\rho,\rho_L}(\alpha) \ := \ (x, x_L) \mapsto \iota_{\rho_L}\Big(n \mapsto \iota_\rho\big(\alpha_{\square,n}\big)(x)\Big)(x_L).$$

We obtain a "multidimensional ladder". By taking (co)limits we can extend the definitions of A_ρ and ι_ρ to $\rho \in \big(\mathbf{R}_{(0,\infty)} \cup \{0, \infty\}\big)^L$.

Expected formulae hold, such as:

Proposition: $\iota_\rho(\alpha)(x) \cdot \iota_\sigma(\beta)(y) = \iota_{\rho,\sigma}\Big((m,n) \mapsto \alpha_m \cdot \beta_n\Big)(x,y)$

Proposition: $\iota_\infty(\alpha)(x+y) = \iota_{\infty,\infty}\Big((m,n) \mapsto \binom{m+n}{m}\alpha_{m+n}\Big)(x,y)$

Corollary: $\exp(x) \cdot \exp(y) = \exp(x+y)$

Also, by using translations, we can evaluate power series around points other than 0.

Concluding remarks

- The proposed axiom allows us to define and work (as classically) with general power series in SDG.
- It implies some other axioms ($\mathbf{R}$ a field, existence of certain elementary functions, the integration axiom for analytical functions).
- **To do**: Which models of SDG satisfy the axiom?
- **Question**: Is there a single general SDG axiom, that would replace all classical limit arguments?