Chow groups of tensor triangulated categories

Sebastian Klein
s.a.klein@uu.nl

Department of Mathematics
Utrecht University

Samuel Eilenberg Centenary Conference 2013

Universiteit Utrecht

Goal of the talk:

For a tensor triangulated category $\mathcal{T}$, define the Chow group $\mathsf{CH}^{\Delta}(\mathcal{T})$ (an abelian group) and show some of its properties.

Outline

Tensor triangular geometry and tensor triangular Chow groups.

- The Chow group of an algebraic variety.
- Tensor triangulated categories.
- Tensor triangular geometry.
- The definition of $\mathsf{CH}^{\Delta}(\mathcal{T})$.

Examples and functoriality properties.

- Examples from algebraic geometry.
- Examples from modular representation theory.
- Functoriality properties of $\mathsf{CH}^{\Delta}(-)$.

The Chow group of an algebraic variety.

- The *codimension p Chow group* of an algebraic variety X, denoted by $\mathrm{CH}^p(X)$, is the free abelian group on the set of codimension p subvarieties of X (the *codimension p cycle group* $\mathrm{Z}^p(X)$) modulo the subgroup of cycles rationally equivalent to zero.
- $\mathrm{CH}(X) := \bigoplus_p \mathrm{CH}^p(X)$ is an interesting invariant of X which can be seen as a generalization of the divisor class group $\mathrm{Div}(X) = \mathrm{CH}^1(X)$. It also plays an important role in the intersection theory of algebraic varieties.
- I want generalize this invariant to tensor triangulated categories.

Universiteit Utrecht

The Chow group of an algebraic variety.

- The *codimension p Chow group* of an algebraic variety X, denoted by $\mathrm{CH}^p(X)$, is the free abelian group on the set of codimension p subvarieties of X (the *codimension p cycle group* $\mathrm{Z}^p(X)$) modulo the subgroup of cycles rationally equivalent to zero.
- $\mathrm{CH}(X) := \bigoplus_p \mathrm{CH}^p(X)$ is an interesting invariant of X which can be seen as a generalization of the divisor class group $\mathrm{Div}(X) = \mathrm{CH}^1(X)$. It also plays an important role in the intersection theory of algebraic varieties.
- I want generalize this invariant to tensor triangulated categories.

The Chow group of an algebraic variety.

- The *codimension p Chow group* of an algebraic variety X, denoted by $\mathsf{CH}^p(X)$, is the free abelian group on the set of codimension p subvarieties of X (the *codimension p cycle group* $\mathsf{Z}^p(X)$) modulo the subgroup of cycles rationally equivalent to zero.
- $\mathsf{CH}(X) := \bigoplus_p \mathsf{CH}^p(X)$ is an interesting invariant of X which can be seen as a generalization of the divisor class group $\mathrm{Div}(X) = \mathsf{CH}^1(X)$. It also plays an important role in the intersection theory of algebraic varieties.
- I want generalize this invariant to tensor triangulated categories.

Universiteit Utrecht

Tensor triangular geometry and tensor triangular Chow groups.

- The Chow group of an algebraic variety.
- Tensor triangulated categories.
- Tensor triangular geometry.
- The definition of $CH^{\Delta}(\mathcal{T})$.

Examples and functoriality properties.

- Examples from algebraic geometry.
- Examples from modular representation theory.
- Functoriality properties of $CH^{\Delta}(-)$.

Tensor triangulated categories.

Definition

A *triangulated category* is an additive category $\mathcal{T}$ equipped with an autoequivalence $\Sigma : \mathcal{T} \to \mathcal{T}$ and a class of *distinguished triangles*, i.e. diagrams of the form

$$A \to B \to C \to \Sigma(A)$$

which satisfy a list of axioms.

Definition

A *tensor triangulated category* is a triangulated category $\mathcal{T}$ equipped with a symmetric monoidal structure

$$- \otimes - : \mathcal{T} \times \mathcal{T} \to \mathcal{T}$$

which is exact in both variables.

Example

Let R be a commutative ring. Then $\mathcal{T} = \mathrm{K}^b(R - \mathrm{proj})$ together with $\otimes = \otimes^L$ is a tensor triangulated category.

Tensor triangulated categories.

Definition

A *triangulated category* is an additive category $\mathcal{T}$ equipped with an autoequivalence $\Sigma : \mathcal{T} \to \mathcal{T}$ and a class of *distinguished triangles*, i.e. diagrams of the form

$$A \to B \to C \to \Sigma(A)$$

which satisfy a list of axioms.

Definition

A *tensor triangulated category* is a triangulated category $\mathcal{T}$ equipped with a symmetric monoidal structure

$$- \otimes - : \mathcal{T} \times \mathcal{T} \to \mathcal{T}$$

which is exact in both variables.

Example

Let R be a commutative ring. Then $\mathcal{T} = \mathrm{K}^b(R - \mathrm{proj})$ together with $\otimes = \otimes^L$ is a tensor triangulated category.

Tensor triangulated categories.

Definition

A *triangulated category* is an additive category $\mathcal{T}$ equipped with an autoequivalence $\Sigma : \mathcal{T} \to \mathcal{T}$ and a class of *distinguished triangles*, i.e. diagrams of the form

$$A \to B \to C \to \Sigma(A)$$

which satisfy a list of axioms.

Definition

A *tensor triangulated category* is a triangulated category $\mathcal{T}$ equipped with a symmetric monoidal structure

$$- \otimes - : \mathcal{T} \times \mathcal{T} \to \mathcal{T}$$

which is exact in both variables.

Example

Let R be a commutative ring. Then $\mathcal{T} = \mathrm{K}^b(R - \mathrm{proj})$ together with $\otimes = \otimes^L$ is a tensor triangulated category.

Thick subcategories and prime tensor ideals.

Convention

From now on, all tensor triangulated categories will be assumed to be essentially small.

Definition

A triangulated subcategory $\mathcal{P} \subset \mathcal{T}$ is

- *thick* if $a \oplus b \in \mathcal{P}$ implies that $a, b \in \mathcal{P}$ for all objects $a, b \in \mathcal{T}$.
- a *tensor ideal* if $\mathcal{T} \otimes \mathcal{P} \subset \mathcal{P}$.
- a *prime ideal* if it is a thick tensor ideal such that $a \otimes b \in \mathcal{P}$ implies that $a \in \mathcal{P}$ or $b \in \mathcal{P}$.

Thick subcategories and prime tensor ideals.

Convention

From now on, all tensor triangulated categories will be assumed to be essentially small.

Definition

A triangulated subcategory $\mathcal{P} \subset \mathcal{T}$ is

- *thick* if $a \oplus b \in \mathcal{P}$ implies that $a, b \in \mathcal{P}$ for all objects $a, b \in \mathcal{T}$.
- a *tensor ideal* if $\mathcal{T} \otimes \mathcal{P} \subset \mathcal{P}$.
- a *prime ideal* if it is a thick tensor ideal such that $a \otimes b \in \mathcal{P}$ implies that $a \in \mathcal{P}$ or $b \in \mathcal{P}$.

Thick subcategories and prime tensor ideals.

Convention

From now on, all tensor triangulated categories will be assumed to be essentially small.

Definition

A triangulated subcategory $\mathcal{P} \subset \mathcal{T}$ is

- *thick* if $a \oplus b \in \mathcal{P}$ implies that $a, b \in \mathcal{P}$ for all objects $a, b \in \mathcal{T}$.
- a *tensor ideal* if $\mathcal{T} \otimes \mathcal{P} \subset \mathcal{P}$.
- a *prime ideal* if it is a thick tensor ideal such that $a \otimes b \in \mathcal{P}$ implies that $a \in \mathcal{P}$ or $b \in \mathcal{P}$.

Universiteit Utrecht

Thick subcategories and prime tensor ideals.

Convention

From now on, all tensor triangulated categories will be assumed to be essentially small.

Definition

A triangulated subcategory $\mathcal{P} \subset \mathcal{T}$ is

- *thick* if $a \oplus b \in \mathcal{P}$ implies that $a, b \in \mathcal{P}$ for all objects $a, b \in \mathcal{T}$.
- a *tensor ideal* if $\mathcal{T} \otimes \mathcal{P} \subset \mathcal{P}$.
- a *prime ideal* if it is a thick tensor ideal such that $a \otimes b \in \mathcal{P}$ implies that $a \in \mathcal{P}$ or $b \in \mathcal{P}$.

The spectrum of a tensor triangulated category and some of its properties.

Definition (Balmer, 2005)

The *spectrum* of $\mathcal{T}$ is the set

$$\mathrm{Spc}(\mathcal{T}) := \{\mathcal{P} \subsetneq \mathcal{T} : \mathcal{P} \text{ is a prime ideal}\}$$

topologized by the basis of closed sets given by

$$\mathrm{supp}(a) := \{\mathcal{P} \in \mathrm{Spc}(\mathcal{T}) : a \notin \mathcal{P}\}$$

for all objects $a \in \mathcal{T}$.

Properties of $\mathrm{Spc}(\mathcal{T})$:

- $\mathrm{Spc}(\mathcal{T})$ is non-empty when $\mathcal{T}$ is non-zero.
- $\mathrm{Spc}(\mathcal{T})$ is always a spectral topological space.
- $\mathrm{Spc}(-)$ behaves functorially w.r.t. tensor exact functors.

The spectrum of a tensor triangulated category and some of its properties.

Definition (Balmer, 2005)

The *spectrum* of $\mathcal{T}$ is the set

$$\mathrm{Spc}(\mathcal{T}) := \{\mathcal{P} \subsetneq \mathcal{T} : \mathcal{P} \text{ is a prime ideal}\}$$

topologized by the basis of closed sets given by

$$\mathrm{supp}(a) := \{\mathcal{P} \in \mathrm{Spc}(\mathcal{T}) : a \notin \mathcal{P}\}$$

for all objects $a \in \mathcal{T}$.

Properties of $\mathrm{Spc}(\mathcal{T})$:

- $\mathrm{Spc}(\mathcal{T})$ is non-empty when $\mathcal{T}$ is non-zero.
- $\mathrm{Spc}(\mathcal{T})$ is always a spectral topological space.
- $\mathrm{Spc}(-)$ behaves functorially w.r.t. tensor exact functors.

The spectrum of a tensor triangulated category and some of its properties.

Definition (Balmer, 2005)

The *spectrum* of $\mathcal{T}$ is the set

$$\mathrm{Spc}(\mathcal{T}) := \{\mathcal{P} \subsetneq \mathcal{T} : \mathcal{P} \text{ is a prime ideal}\}$$

topologized by the basis of closed sets given by

$$\mathrm{supp}(a) := \{\mathcal{P} \in \mathrm{Spc}(\mathcal{T}) : a \notin \mathcal{P}\}$$

for all objects $a \in \mathcal{T}$.

Properties of $\mathrm{Spc}(\mathcal{T})$:

- $\mathrm{Spc}(\mathcal{T})$ is non-empty when $\mathcal{T}$ is non-zero.
- $\mathrm{Spc}(\mathcal{T})$ is always a spectral topological space.
- $\mathrm{Spc}(-)$ behaves functorially w.r.t. tensor exact functors.

The spectrum of a tensor triangulated category and some of its properties.

Definition (Balmer, 2005)

The *spectrum* of $\mathcal{T}$ is the set

$$\mathrm{Spc}(\mathcal{T}) := \{\mathcal{P} \subsetneq \mathcal{T} : \mathcal{P} \text{ is a prime ideal}\}$$

topologized by the basis of closed sets given by

$$\mathrm{supp}(a) := \{\mathcal{P} \in \mathrm{Spc}(\mathcal{T}) : a \notin \mathcal{P}\}$$

for all objects $a \in \mathcal{T}$.

Properties of $\mathrm{Spc}(\mathcal{T})$:

- $\mathrm{Spc}(\mathcal{T})$ is non-empty when $\mathcal{T}$ is non-zero.
- $\mathrm{Spc}(\mathcal{T})$ is always a spectral topological space.
- $\mathrm{Spc}(-)$ behaves functorially w.r.t. tensor exact functors.

Universiteit Utrecht

Tensor triangular geometry and tensor triangular Chow groups.

- The Chow group of an algebraic variety.
- Tensor triangulated categories.
- Tensor triangular geometry.
- The definition of $\mathrm{CH}^{\Delta}(\mathcal{T})$.

Examples and functoriality properties.

- Examples from algebraic geometry.
- Examples from modular representation theory.
- Functoriality properties of $\mathrm{CH}^{\Delta}(-)$.

Tensor triangular cycle groups.

For $p \in \mathbb{Z}_{\geq 0}$, define the full triangulated subcategory

$$\mathcal{T}_{(p)} := \{a \in \mathcal{T} : \mathrm{codim}_{\mathrm{Krull}}(\mathrm{supp}(a)) \geq p\}.$$

This gives a filtration of $\mathcal{T}$ by codimension of support:

$$\mathcal{T} = \mathcal{T}_{(0)} \supset \mathcal{T}_{(1)} \supset \mathcal{T}_{(2)} \supset \cdots$$

Definition (Balmer)

For $p \in \mathbb{Z}_{\geq 0}$, the *codimension p tensor triangular cycle group* of $\mathcal{T}$ is defined as

$$\mathrm{Z}_p^{\Delta}(\mathcal{T}) := \mathrm{K}_0\left(\left(\mathcal{T}_{(p)}/\mathcal{T}_{(p+1)}\right)^{\natural}\right)$$

where $\left(\mathcal{T}_{(p)}/\mathcal{T}_{(p+1)}\right)^{\natural}$ denotes the idempotent completion of the Verdier quotient $\mathcal{T}_{(p)}/\mathcal{T}_{(p+1)}$.

Tensor triangular cycle groups.

For $p \in \mathbb{Z}_{\geq 0}$, define the full triangulated subcategory

$$\mathcal{T}_{(p)} := \{a \in \mathcal{T} : \mathrm{codim}_{\mathrm{Krull}}(\mathrm{supp}(a)) \geq p\}.$$

This gives a filtration of $\mathcal{T}$ by codimension of support:

$$\mathcal{T} = \mathcal{T}_{(0)} \supset \mathcal{T}_{(1)} \supset \mathcal{T}_{(2)} \supset \cdots$$

Definition (Balmer)

For $p \in \mathbb{Z}_{\geq 0}$, the *codimension p tensor triangular cycle group* of $\mathcal{T}$ is defined as

$$\mathsf{Z}_p^{\Delta}(\mathcal{T}) := \mathsf{K}_0\left(\left(\mathcal{T}_{(p)}/\mathcal{T}_{(p+1)}\right)^{\natural}\right)$$

where $\left(\mathcal{T}_{(p)}/\mathcal{T}_{(p+1)}\right)^{\natural}$ denotes the idempotent completion of the Verdier quotient $\mathcal{T}_{(p)}/\mathcal{T}_{(p+1)}$.

Definition of $\mathrm{CH}_p^{\Delta}(\mathcal{T})$.

Consider the following diagram:

$$\begin{array}{ccccccc}
\mathrm{K}_0(\mathcal{T}_{(p)}) & \xrightarrow{p} & \mathrm{K}_0\left(\mathcal{T}_{(p)}/\mathcal{T}_{(p+1)}\right) & \xrightarrow{j} & \mathrm{K}_0\left(\left(\mathcal{T}_{(p)}/\mathcal{T}_{(p+1)}\right)^{\natural}\right) & = & \mathrm{Z}_p^{\Delta}(\mathcal{T}) \\
\big\downarrow{\scriptstyle i} & & & & & & \\
\mathrm{K}_0(\mathcal{T}_{(p-1)}) & & & & & &
\end{array}$$

Definition (Balmer)

The *codimension p tensor triangular Chow group* of $\mathcal{T}$ is defined as

$$\mathrm{CH}_p^{\Delta}(\mathcal{T}) := \mathrm{Z}_p^{\Delta}(\mathcal{T})/j \circ p(\ker(i)).$$

Definition of $\mathrm{CH}_p^{\Delta}(\mathcal{T})$.

Consider the following diagram:

$$\begin{array}{ccccccc}
\mathrm{K}_0(\mathcal{T}_{(p)}) & \xrightarrow{p} & \mathrm{K}_0\left(\mathcal{T}_{(p)}/\mathcal{T}_{(p+1)}\right) & \xrightarrow{j} & \mathrm{K}_0\left(\left(\mathcal{T}_{(p)}/\mathcal{T}_{(p+1)}\right)^{\natural}\right) & = & \mathrm{Z}_p^{\Delta}(\mathcal{T}) \\
\downarrow{\scriptstyle i} & & & & & & \\
\mathrm{K}_0(\mathcal{T}_{(p-1)}) & & & & & &
\end{array}$$

Definition (Balmer)

The *codimension p tensor triangular Chow group* of $\mathcal{T}$ is defined as

$$\mathrm{CH}_p^{\Delta}(\mathcal{T}) := \mathrm{Z}_p^{\Delta}(\mathcal{T})/j \circ p(\ker(i)).$$

Tensor triangular geometry and tensor triangular Chow groups.

- The Chow group of an algebraic variety.
- Tensor triangulated categories.
- Tensor triangular geometry.
- The definition of $\mathrm{CH}^{\Delta}(\mathcal{T})$.

Examples and functoriality properties.

- Examples from algebraic geometry.
- Examples from modular representation theory.
- Functoriality properties of $\mathrm{CH}^{\Delta}(-)$.

The derived category of perfect complexes.

- For an algebraic variety X, consider the category

$$\mathsf{D}^{\mathsf{perf}}(X) \subset \mathsf{D}^{\mathsf{b}}(\mathrm{Coh}(X)).$$

  This is an essentially small tensor triangulated category with tensor product $\otimes^L$.
- $\mathrm{Spc}(\mathsf{D}^{\mathsf{perf}}(X)) \cong X$ (Balmer, 2005).
- If X is regular $\mathsf{D}^{\mathsf{perf}}(X) \cong \mathsf{D}^{\mathsf{b}}(\mathrm{Coh}(X))$.

The derived category of perfect complexes.

- For an algebraic variety X, consider the category

$$\mathsf{D}^{\mathsf{perf}}(X) \subset \mathsf{D}^{\mathsf{b}}(\mathrm{Coh}(X)).$$

  This is an essentially small tensor triangulated category with tensor product $\otimes^L$.
- $\mathrm{Spc}(\mathsf{D}^{\mathsf{perf}}(X)) \cong X$ (Balmer, 2005).
- If X is regular $\mathsf{D}^{\mathsf{perf}}(X) \cong \mathsf{D}^{\mathsf{b}}(\mathrm{Coh}(X))$.

The derived category of perfect complexes.

- For an algebraic variety X, consider the category

$$\mathsf{D}^{\mathsf{perf}}(X) \subset \mathsf{D}^{\mathsf{b}}(\mathrm{Coh}(X)).$$

  This is an essentially small tensor triangulated category with tensor product $\otimes^L$.
- $\mathsf{Spc}(\mathsf{D}^{\mathsf{perf}}(X)) \cong X$ (Balmer, 2005).
- If X is regular $\mathsf{D}^{\mathsf{perf}}(X) \cong \mathsf{D}^{\mathsf{b}}(\mathrm{Coh}(X))$.

Comparison theorem.

Theorem (Balmer, K.)

Let X be a regular variety. For all $p \in \mathbb{Z}_{\geq 0}$ there are isomorphisms

$$\mathrm{Z}_p^{\Delta}\left(\mathrm{D}^{\mathrm{perf}}(X)\right) \cong \mathrm{Z}^p(X)$$

and

$$\mathrm{CH}_p^{\Delta}\left(\mathrm{D}^{\mathrm{perf}}(X)\right) \cong \mathrm{CH}^p(X)$$

where $\mathrm{CH}^p(X)$ and $\mathrm{Z}^p(X)$ are the usual codimension p Chow group and cycle group of X.

Modular representation theory.

- Let k be a field of characteristic p, G a finite group such that p divides $\#G$.
- In modular representation theory, one studies the category kG-mod of finite-dimensional kG-left modules (k-representations of G) and the associated stable category kG-stab.
- The category kG-stab is an essentially small tensor triangulated category. The tensor structure is given by the tensor product of modules $\otimes_k$ *over* k.
- $\mathrm{Spc}(kG\text{-stab}) \cong \mathcal{V}_G(k)$, the *projective support variety* of kG.

Modular representation theory.

- Let k be a field of characteristic p, G a finite group such that p divides $\#G$.
- In modular representation theory, one studies the category kG-mod of finite-dimensional kG-left modules (k-representations of G) and the associated stable category kG-stab.
- The category kG-stab is an essentially small tensor triangulated category. The tensor structure is given by the tensor product of modules $\otimes_k$ *over* k.
- $\mathrm{Spc}(kG\text{-stab}) \cong \mathcal{V}_G(k)$, the *projective support variety* of kG.

Modular representation theory.

- Let k be a field of characteristic p, G a finite group such that p divides $\#G$.
- In modular representation theory, one studies the category kG-mod of finite-dimensional kG-left modules (k-representations of G) and the associated stable category kG-stab.
- The category kG-stab is an essentially small tensor triangulated category. The tensor structure is given by the tensor product of modules $\otimes_k$ *over* k.
- $\mathrm{Spc}(kG\text{-stab}) \cong \mathcal{V}_G(k)$, the *projective support variety* of kG.

Modular representation theory.

- Let k be a field of characteristic p, G a finite group such that p divides $\#G$.
- In modular representation theory, one studies the category kG-mod of finite-dimensional kG-left modules (k-representations of G) and the associated stable category kG-stab.
- The category kG-stab is an essentially small tensor triangulated category. The tensor structure is given by the tensor product of modules $\otimes_k$ *over* k.
- $\mathrm{Spc}(kG\text{-stab}) \cong \mathcal{V}_G(k)$, the *projective support variety* of kG.

Example: Cyclic groups and V_4.

Example

Let k be a field of characteristic p and $G = \mathbb{Z}/p^n\mathbb{Z}$ for some $n \geq 1$. Then $\mathcal{V}_G(k) = \{*\}$ and

$$\mathrm{Z}_0^{\Delta}(kG\text{-stab}) = \mathrm{CH}_0^{\Delta}(kG\text{-stab}) \cong \mathbb{Z}/p^n\mathbb{Z}.$$

Note that we get cycle groups with torsion!

Example (K.)

Let k be a an algebraically closed field of characteristic 2 and $G = V_4$. Then $\mathcal{V}_G(k) = \mathbb{P}^1_k$ and

$$\mathrm{CH}^{\Delta}(kG\text{-stab}) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$

Note that in both examples $\mathrm{CH}^{\Delta}(kG\text{-stab}) \neq \mathrm{CH}(\mathcal{V}_G(k))$.

Example: Cyclic groups and V_4.

Example

Let k be a field of characteristic p and $G = \mathbb{Z}/p^n\mathbb{Z}$ for some $n \geq 1$. Then $\mathcal{V}_G(k) = \{*\}$ and

$$\mathrm{Z}_0^{\Delta}(kG\text{-stab}) = \mathrm{CH}_0^{\Delta}(kG\text{-stab}) \cong \mathbb{Z}/p^n\mathbb{Z}.$$

Note that we get cycle groups with torsion!

Example (K.)

Let k be a an algebraically closed field of characteristic 2 and $G = V_4$. Then $\mathcal{V}_G(k) = \mathbb{P}^1_k$ and

$$\mathrm{CH}^{\Delta}(kG\text{-stab}) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$

Note that in both examples $\mathrm{CH}^{\Delta}(kG\text{-stab}) \neq \mathrm{CH}(\mathcal{V}_G(k))$.

Example: Cyclic groups and V_4.

Example

Let k be a field of characteristic p and $G = \mathbb{Z}/p^n\mathbb{Z}$ for some $n \geq 1$. Then $\mathcal{V}_G(k) = \{*\}$ and

$$Z_0^{\Delta}(kG\text{-stab}) = CH_0^{\Delta}(kG\text{-stab}) \cong \mathbb{Z}/p^n\mathbb{Z}.$$

Note that we get cycle groups with torsion!

Example (K.)

Let k be a an algebraically closed field of characteristic 2 and $G = V_4$. Then $\mathcal{V}_G(k) = \mathbb{P}^1_k$ and

$$CH^{\Delta}(kG\text{-stab}) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$

Note that in both examples $CH^{\Delta}(kG\text{-stab}) \neq CH(\mathcal{V}_G(k))$.

Example: Cyclic groups and V_4.

Example

Let k be a field of characteristic p and $G = \mathbb{Z}/p^n\mathbb{Z}$ for some $n \geq 1$. Then $\mathcal{V}_G(k) = \{*\}$ and

$$\mathrm{Z}_0^{\Delta}(kG\text{-stab}) = \mathrm{CH}_0^{\Delta}(kG\text{-stab}) \cong \mathbb{Z}/p^n\mathbb{Z}.$$

Note that we get cycle groups with torsion!

Example (K.)

Let k be a an algebraically closed field of characteristic 2 and $G = V_4$. Then $\mathcal{V}_G(k) = \mathbb{P}_k^1$ and

$$\mathrm{CH}^{\Delta}(kG\text{-stab}) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$

Note that in both examples $\mathrm{CH}^{\Delta}(kG\text{-stab}) \neq \mathrm{CH}(\mathcal{V}_G(k))$.

Tensor triangular geometry and tensor triangular Chow groups.

- The Chow group of an algebraic variety.
- Tensor triangulated categories.
- Tensor triangular geometry.
- The definition of $\mathrm{CH}^{\Delta}(\mathcal{T})$.

Examples and functoriality properties.

- Examples from algebraic geometry.
- Examples from modular representation theory.
- Functoriality properties of $\mathrm{CH}^{\Delta}(-)$.

Universiteit Utrecht

Functoriality and derived flat pull-back/proper push-forward.

Lemma

An exact functor between tensor triangulated categories that has a relative dimension induces homomorphisms of tensor triangular cycle groups and Chow groups.

Theorem (K.)

If f is a flat (resp. proper) morphism of regular algebraic varieties then Lf^* (resp. Rf_*) have a relative dimension and the induced homomorphisms on tensor triangular cycle groups and Chow groups is the expected flat pull-back (resp. proper push-forward).

Functoriality and derived flat pull-back/proper push-forward.

Lemma

An exact functor between tensor triangulated categories that has a relative dimension induces homomorphisms of tensor triangular cycle groups and Chow groups.

Theorem (K.)

If f is a flat (resp. proper) morphism of regular algebraic varieties then Lf^* (resp. Rf_*) have a relative dimension and the induced homomorphisms on tensor triangular cycle groups and Chow groups is the expected flat pull-back (resp. proper push-forward).

Summary

- For a sufficiently nice tensor triangulated category $\mathcal{T}$ we can define its Chow group $\mathrm{CH}^{\Delta}(\mathcal{T})$ that recovers $\mathrm{CH}(X)$ when X is a non-singular algebraic variety and $\mathcal{T} = \mathrm{D}^{\mathrm{perf}}(X)$.
- $\mathrm{CH}^{\Delta}(-)$ behaves functorially for a large class of functors and also gives non-trivial results for tensor triangulated categories that don't come from algebraic geometry.

Outlook: I'm currently working on an intersection product for the tensor triangulated Chow groups. This will require the use of a suitable model for the category $\mathcal{T}$ in order to be able to work with the higher K-theory of $\mathcal{T}$.