Simplicial sets in Perspective

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The EZ paper

The notion of *simplicial set* was introduced by

Samuel Eilenberg and J.A. Zilber

in a paper called

Semi-simplicial Complexes and Singular Homology

published in the Annals of Mathematics (1950)

The EZ paper(2)

The EZ paper refers to 6 papers

- S. Eilenberg, *Singular homology theory*, Ann. of Math. 45 (1944):
- S. Eilenberg, *Topological methods in abstract algebra; Cohomology theory of abstract groups*, Bull. AMS (1949)
- S. Eilenberg and S. MacLane, *Cohomology theory in abstract groups. I*, Ann. of Math. (1947)
- S. Eilenberg and S. MacLane, *Relations between homology and homotopy groups of spaces. II*, Ann. of Math. (1950)
- N. E. Steenrod, *Homology with local coefficients*, Ann. of Math. (1943)
- N. E. Steenrod, *Products of cocycles and extensions of mappings*, Ann. of Math. (1947)

Simplicial complexes

In the EZ paper,

A simplicial set is called a *complete semi-simplicial complex*. We recall that

Definition

A *simplicial complex* is a set X equipped with a set $S(X)$ of non-empty subsets of X such that

$$\emptyset \neq A \subseteq B \in S(X) \Longrightarrow A \in S(X)$$

An element of $S(X)$ is called a *simplex* of X.

The category Δ

The category Δ was introduced in the EZ-paper,

Δ is the category of finite non-empty ordinals $[n] = \{0, \ldots, n\}$ $(n \geq 0)$ and order preserving maps $f : [m] \to [n]$:

$$x \leq y \Rightarrow f(x) \leq f(y).$$

- $d_i : [n-1] \to [n]$ is the injection omitting $i \in [n]$
- $s_i : [n+1] \to [n]$ is the surjection repeating $i \in [n]$

The simplicial identities are observed:

- $d_i d_j = d_{j-1} d_i$ for $i < j$
- $s_i s_j = s_{j+1} s_i$ for $i \leq j$
- $d_i s_j = s_{j-1} d_i$ for $i < j$
- $d_i s_i = id = d_{i+1} s_i$
- $d_i s_j = s_j d_{i-1}$ for $i > j+1$

Simplicial sets

The notion of *simplicial set* is introduced in the EZ-paper.

Definition

- A *semi-simplicial complex* is a contravariant functor $X : \Delta^{mono} \to \mathbf{Set}$.
- A *complete semi-simplicial complex* (ie a **simplicial set**) is a contravariant functor $X : \Delta \to \mathbf{Set}$.
- A *map* of (complete) semi-simplicial complexes $f : X \to Y$ is a natural transformation.

Remark: EZ are not using the language of category theory!

Notation:

- $X_n = X([n])$
- $\partial_i = X(d_i) : X_n \to X_{n-1}$
- $\sigma_i = X(s_i) : X_n \to X_{n+1}$

The singular complex

In the EZ paper, the notion of simplicial set is motivated by the example of the the singular complex $S(X)$ of a topological space X.

Recall that $S(X) : \Delta^{op} \to \mathbf{Set}$ is defined by putting

$$S(X)_n = Map(\Delta_n, X)$$

where

$$\Delta_n = \{(t_0, \ldots, t_n) : \sum_i t_i = 1, t_i \geq 0\}$$

is the *geometric simplex* of dimension n.

The nerve of a group

The nerve of a group G is another example of simplicial set.

Recall that $N(G) : \Delta^{op} \to \mathbf{Set}$ defined by putting $N(G)_n = G^n$ and

$$\partial_i(g_1, \dots, g_n) = \begin{cases} (g_2, \dots, g_n) & \text{if} \quad i = 0 \\ (g_1, \dots, g_{i-1}g_i, \dots, g_n) & \text{if} \quad 0 < i < n \\ (g_1, \dots, g_{n-1}) & \text{if} \quad i = n \end{cases}$$

$$\sigma_i(g_1, \dots, g_n) = \begin{cases} (1, g_1, \dots, g_n) & \text{if} \quad i = 0 \\ (g_1, \dots, g_i, 1, g_{i+1}, \dots, g_n) & \text{if} \quad 0 < i < n \\ (g_1, \dots, g_n, 1) & \text{if} \quad i = n \end{cases}$$

Homology of a simplicial set

Recall that a simplicial set X has *homology groups*

$$H_n(X) = H_n(C_\star(X)),$$

where $C_\star(X)$ is the chain complex associated to X:

$$C_0(X) \xleftarrow{\partial_1} C_1(X) \xleftarrow{\partial_2} C_2(X) \xleftarrow{\partial_3} C_3(X) \longleftarrow \cdots$$

By construction

- $C_n X = \mathbb{Z} X_n$
- $\partial_n = \sum_{i=0}^{n} (-1)^i \partial_i$

Singular homology

Definition

(**Eilenberg**) The *singular homology* of a space X is the homology of its singular complex $S(X)$.

Theorem

*(**Eilenberg**) The homology of a finite simplicial complex is isomorphic to the singular homology of its geometric realization.*

S. Eilenberg, *Singular homology theory*, Ann. of Math. 45 (1944):

Note: A notion of singular homology based on *oriented* singular simplices had been introduced previously by Lefschetz (1933).

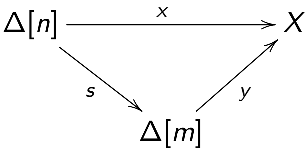
Eilenberg-Zilber theorem

Definition

A simplex $x \in X_n$ is *degenerate* if it belongs to the image of a degeneracy map $\sigma_i : X_{n-1} \to X_n$.

Theorem

(**Eilenberg-Zilber**) *For every simplex $x \in X_n$ there exists a non-degenerate simplex $y \in X_m$ together with a surjective map $s : [n] \to [m]$ such that $ys = x$. Moreover, the pair (y, s) is unique.*



Normalisation

A cochain $f : C_n(X) \to G$ is *normalised* if $f(x) = 0$ for every degenerate simplex $x \in X_n$.

Theorem

*(**Eilenberg-Zilber**) The canonical map $H^\star_N(X, G) \to H^\star(X, G)$ is invertible for any abelian group G.*

This extend a normalization theorem proved by EM in the cohomology theory of groups:

S. Eilenberg and S. MacLane, *Cohomology theory in abstract groups. I*, Ann. of Math. (1947)

Minimal complexes

Eilenberg and Zilber:

Although the singular complex $S(X)$ of a space X is very "large" it is possible to find subcomplexes of $S(X)$ which contains all the information that $S(X)$ carries but which are stripped of everything superfluous from the point of view of homotopy. The existence and uniqueness of such minimal complexes is established. They are the main tool in the paper of Eilenberg and MacLane immediately following:

S. Eilenberg and S. MacLane, *Relations between homology and homotopy groups of spaces. II*, Ann. of Math. (1950)

Combinatorial homotopy theory

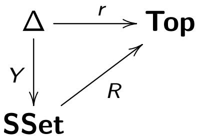
Daniel Kan:

- *Adjoint functors*, Proc. of the Nat. Acad. of Sciences. (1956)
- *Abstract Homotopy. III*, Proc. of the Nat. Acad. Sc. (1956)
- *On c.s.s. complexes*, American Jour. of Math. (1957)
- *Functors involving c.s.s. complexes*, Trans, AMS (1958)

Geometric realisation:

$$R : \mathbf{SSet} \leftrightarrow \mathbf{Top} : S$$

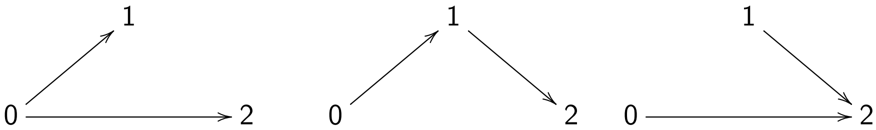
Kan extensions:



Horns

Horns:

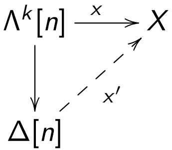
$$\Lambda^k[n] = \bigcup_{i \neq k} \partial_i \Delta[n]$$



$\Lambda^0[2]$ $\Lambda^1[2]$ $\Lambda^2[2]$

Kan complexes

Recall that a simplicial set X is a *Kan complex* if every horn $\Lambda^k[n] \to X$ has a filler $x' : \Delta[n] \to X$.



Theorem

*(**Kan**) The homotopy category of Kan complexes is equivalent to the homotopy category of CW-complexes*

The Dold-Kan correspondance

Dold: *Homology of Symmetric Products and Other Functors of Complexes*, (1957)

Kan:*Functors involving css complexes*, (1958)

The Dold-Kan correspondence is an equivalence

$$N : [\Delta^{op}, \mathbf{Ab}] \leftrightarrow Ch^+(\mathbf{Ab}) : \Gamma$$

between the category of simplicial abelian groups and the category of (non-negative) chain complexes.

The functor N takes a simplicial abelian group to its *normalized* chain complex. The functor Γ takes a chain complex $A = (A, \partial)$ to the simplicial abelian group $\Gamma(A)$ defined by putting

$$\Gamma(A)_n = Hom(N(\mathbb{Z}\Delta[n]), A)$$

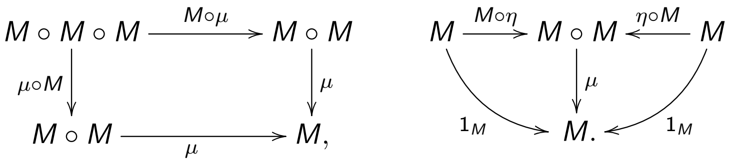
for every $n \geq 0$.

Constructions standards

Godement: *Topologie algébrique et théorie des faisceaux* (1958)

A **monad** on a category $\mathcal{C}$ is a triple (M, μ, η) where

- M is an endofunctor $\mathcal{C} \to \mathcal{C}$;
- $\mu : M \circ M \to M$ and $\eta : Id \to M$ are natural transformations;
- the following diagrams commutes:



Monads from adjunctions

Let $F : \mathcal{C} \leftrightarrow \mathcal{E} : U$ be an adjunction, with unit $\eta : I \to UF$ and counit $\epsilon : FU \to I$.

Let us put $M = U \circ F$ and $\mu = U \circ \epsilon \circ F$.

The triple (M, μ, η) is a monad.

Conversely

Theorem

(**Eilenberg-Moore**) *Every monad (M, μ, η) in a category $\mathcal{C}$ is associated to an adjunction $F : \mathcal{C} \leftrightarrow \mathcal{C}^M : U$ where $\mathcal{C}^M$ is the category of M-algebras.*

Comonads

There is a dual notion of a **comonad** (C, δ, ϵ).

From an adjunction $F : \mathcal{E} \leftrightarrow \mathcal{F} : U$ we obtain a comonad $C = F \circ U$.

For every object $A \in \mathcal{F}$ let us put:

- $R(A)_n = C^{n+1}(A)$
- $\partial_i = C^i \circ \epsilon \circ C^{n-i}$
- $\sigma_i = C^i \circ \delta \circ C^{n-i-1}$

This defines a simplicial object $R(A)$.

The map $\epsilon : FU(A) \to A$ is an *augmentation* $R(A)_0 \to A$.

The simplical object $R(A)$ is a **free resolution** of A.

The universal monoid

Let **Ord** be the category whose objects are finite ordinals $\underline{n} = \{1, \ldots, n\}$ for $n \geq 0$.

The ordinal sum $\underline{m} + \underline{n} = \underline{m+n}$ gives the category **Ord** the structure of a monoidal category.

The (unique) maps $\mu : \underline{1} + \underline{1} \to \underline{1}$ and $\eta : \underline{0} \to \underline{1}$ exhibit a monoid structure on the object $\underline{1}$.

Theorem

*(**Lawvere**) If $A = (A, \mu, \eta)$ is a monoid in a monoidal category $\mathcal{E} = (\mathcal{E}, \otimes, I)$, then there exists a monoidal functor*

$$F : \mathbf{Ord} \to \mathcal{E}$$

such that $F(\underline{1}) = A$, and F is essentially unique.

The bar complex

Remark:

If $D : \mathbf{Ord}^{op} \to \mathbf{Ord}$ is the *dualising functor* defined by putting

$$D(\underline{n}) = Hom(\underline{n}, \underline{2})$$

then the composite

$$\mathbf{Ord}^{op} \xrightarrow{D} \mathbf{Ord} \xrightarrow{F} \mathcal{E}$$

is an augmented simplicial object $\Delta_{+}^{op} \to \mathcal{E}$.

It is the free resolution of A as an (A, A)-bimodule.

It is the (free) *bar-complex* of A.

It is the *Hochschild complex* of A.

Higher categories and simplicial sets

The main models of $(\infty, 1)$-categories are:

- Simplicial categories (**Dwyer-Kan-Bergner**);
- Segal categories (**Hirshowitz-Simpson**);
- Complete Segal spaces (**Rezk**);
- Quasi-categories (**Boardman-Vogt-J**).

They are constructed by using simplicial sets.

Homotopical algebraic geometry

Bertrand Töen, **Gabriele Vezzosi** and **Jacob Lurie**

What is a (derived) algebraic stack?

- (**Rezk**) A *higher topos* is a homotopy left exact localization of a category of simplicial presheaves on a simplicial site;
- A *pre-stack* is a simplicial pre-sheaf on a simplicial site;
- For derived algebraic geometry, the site is the category of (small) commutative simplicial rings $\mathcal{R}$;
- The localization is defined by a Grothendieck topology on the homotopy category of $\mathcal{R}$;
- A (derived) stack is a fibrant object of the localised model category.

Schemes $\rightarrow$ Deligne-Mumford stacks $\rightarrow$ Derived Algebraic stack

(∞, n)-categories

Pellisier and Simpson:
n-fold Segal categories are the fibrant objects of a model structure on an n-fold simplicial sets.

Rezk:
Complete Segal n-spaces are the fibrant objects of a model structure on the category of simplicial objects in $\Theta_n\mathbf{Set}$.

Ara:
Higher n-quasi-categories are fibrant objects of a model structure on $\Theta_n\mathbf{Set}$.

But what is a $\Theta_n\mathbf{Set}$?

1-Disks

A 1-*disk* is an interval.

An *interval* is a linearly ordered set I with a first and a last elements, say 0 and 1.

An interval I is *non-degenerate* if $0 < 1$, that is, if $\mathrm{Card}(I) > 1$.

Examples,

- $I = [a, b] \subseteq \mathbb{R}$ if $a \leq b$,
- $I = \{1, \ldots, n\}$ if $n \geq 1$,
- The 1-simplex $\Delta[1]$.

The 1-simplex $\Delta[1]$ is an interval in the caterory **SSet**; the ordering is given by the 2-simplex : $\Delta[2] \subset \Delta[1] \times \Delta[1]$.

Duality

A *morphism of intervals* $f : I \to J$ is an order preserving map such that $f(0) = 0$ and $f(1) = 1$.

There is a perfect duality

$$D : \mathbf{Ord} \leftrightarrow \mathbf{Int} : D$$

between the category of finite ordinals **Ord** and the category of finite intervals **Int**,

$$D(\underline{n}) = \mathbf{Ord}(\underline{n}, \underline{2}) \qquad \text{and} \qquad D(\underline{n}) = \mathbf{Int}(\underline{n}, \underline{2}).$$

The duality matches the category of finite *non-degenerate* intervals with the category of finite *non-empty* ordinals Δ.

SSet as a classifying topos

The interval $\Delta[1]$ is *universal*:

Theorem

(**MacLane, Moerdijk**) *The topos* **SSet** *classifies (non-degenerate) intervals.*

Thus, if I is a non-degenerate interval in a topos $\mathcal{E}$, then there exists a geometric homomorphism $h : \mathbf{SSet} \to \mathcal{E}$ such that $h(\Delta[1]) \simeq I$, and h is essentially unique.

The geometric morphism $h : \mathbf{SSet} \to \mathcal{E}$ is a *geometric realisation functor*.

2-Disks

Definition

A **2-disk** is set D equipped with a map $p : D \to I$ having the following structure:

- I is an interval;
- each fiber $D(i) = p^{-1}(i)$ has the structure of an interval;
- the interval $D(i)$ degenerates exactly when $i \in \partial I$.

Example: $D = \{x \in \mathbb{R}^2 : ||x||^2 \leq 1\}$, $I = [-1, 1]$ and $p = p_1$.

Boundary:

$$\partial D = s(I) \cup t(I),$$

where $s : I \to D$ and $t : I \to D$ are lower and upper sections.

A disk D is *degenerate* if D=$\{\star\}$.

n-Disks

Definition

A n-**disk** is set D equipped a map $f : D \to D_{n-1}$ having the following structure:

- D_{n-1} has the structure of a $(n-1)$-disk;
- the fiber $f^{-1}(i)$ has the structure of an interval;
- the interval $f^{-1}(i)$ degenerates exactly when $i \in \partial D_{n-1}$.

Boundary

$$\partial D = s(D_{n-1}) \cup t(D_{n-1}),$$

where $s : D_{n-1} \to D$ and $t : D_{n-1} \to D$ are the lower and upper sections.

A n-disk D is *degenerate* if $D = \{\star\}$.

Θ(n)-Sets

There is a category $\mathcal{D}(n)$ of n-disks

Duality:

Theorem

*(**Makkai-Zawadovski**) The category $\Theta_n = \mathcal{D}(n)^{op}$ is a full subcategory of the category of strict n-categories.*

By definition,

$$\Theta_n\mathbf{Set} = [\Theta_n^{op}, \mathbf{Set}] = [\mathcal{D}(n), \mathbf{Set}].$$

Classifying topos for n-disks

The forgetful functor $\mathcal{D}(n) \to \mathbf{Set}$ has the structure of a n-disk $D[n]$ in $\Theta_n\mathbf{Set}$.

Theorem

*(**J**) The topos $\Theta_n\mathbf{Set}$ classifies (non-degenerate) n-disks.*

Thus, if D is a (non-degenerate) n-disk in a topos $\mathcal{E}$, then there exists a geometric homomorphism $h : \Theta_n\mathbf{Set} \to \mathcal{E}$ such that $h(D[n]) \simeq D$, and h is essentially unique.

The geometric morphism $h : \Theta_n\mathbf{Set} \to \mathcal{E}$ is a *geometric realisation functor*.

Epilogue

Simplicial sets are fundamental objects of mathematics

Thank you Sammy!