Cohomology and crossed products for weak Hopf algebras

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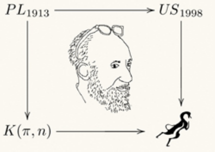

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Outline

Some notation and conventions.

- From now on $\mathcal{C}$ denotes a strict symmetric category with tensor product denoted by $\otimes$ and unit object K. With c we will denote the natural isomorphism of symmetry and we also assume that every idempotent morphism $q : Y \rightarrow Y$ splits, i.e., there exist an object Z and morphisms $i : Z \rightarrow Y$ and $p : Y \rightarrow Z$ such that $q = i \circ p$ and $p \circ i = id_Z$.

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- $(C, \varepsilon_C, \delta_C)$ is a coassociative coalgebra (comonoid) with comultiplication δ_C and counit ε_C.
- For simplicity of notation, given three objects V, U, B in $\mathcal{C}$ and a morphism $f : V \to U$, we write

$$B \otimes f \text{ for } id_B \otimes f \text{ and } f \otimes B \text{ for } f \otimes id_B.$$

The Sweedler cohomology in a weak setting

Definition.

A weak Hopf algebra in $\mathcal{C}$ is an object in $\mathcal{C}$ with an algebra structure (H, η_H, μ_H) and a coalgebra structure $(H, \varepsilon_H, \delta_H)$ satisfying:

(a1) $\delta_H \circ \mu_H = (\mu_H \otimes \mu_H) \circ (H \otimes c_{H,H} \otimes H) \circ (\delta_H \otimes \delta_H)$.

(a2) $\varepsilon_H \circ \mu_H \circ (\mu_H \otimes H) = ((\varepsilon_H \circ \mu_H) \otimes (\varepsilon_H \circ \mu_H)) \circ (H \otimes \delta_H \otimes H)$
$= ((\varepsilon_H \circ \mu_H) \otimes (\varepsilon_H \circ \mu_H)) \circ (H \otimes (c_{H,H} \circ \delta_H) \otimes H)$.

(a3) $(\delta_H \otimes H) \circ \delta_H \circ \eta_H = (H \otimes \mu_H \otimes H) \circ ((\delta_H \circ \eta_H) \otimes (\delta_H \circ \eta_H))$
$= (H \otimes (\mu_H \circ c_{H,H}) \otimes H) \circ ((\delta_H \circ \eta_H) \otimes (\delta_H \circ \eta_H))$.

(a4) There exists a morphism $\lambda_H : H \to H$ in $\mathcal{C}$ (called the antipode of H) satisfying:

- (a4-1) $id_H \wedge \lambda_H = ((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes c_{H,H}) \circ ((\delta_H \circ \eta_H) \otimes H)$.
- (a4-2) $\lambda_H \wedge id_H = (H \otimes (\varepsilon_H \circ \mu_H)) \circ (c_{H,H} \otimes H) \circ (H \otimes (\delta_H \circ \eta_H))$.
- (a4-3) $\lambda_H \wedge id_H \wedge \lambda_H = \lambda_H$.

If H is a weak Hopf algebra in $\mathcal{C}$, the antipode λ_H is unique, antimultiplicative, anticomultiplicative and leaves the unit and the counit invariant:

$$\lambda_H \circ \mu_H = \mu_H \circ (\lambda_H \otimes \lambda_H) \circ c_{H,H}, \quad \delta_H \circ \lambda_H = c_{H,H} \circ (\lambda_H \otimes \lambda_H) \circ \delta_H,$$

$$\lambda_H \circ \eta_H = \eta_H, \quad \varepsilon_H \circ \lambda_H = \varepsilon_H.$$

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$$\lambda_H \circ \eta_H = \eta_H, \quad \varepsilon_H \circ \lambda_H = \varepsilon_H.$$

If we define the morphisms Π_H^L (target), Π_H^R (source), by

$$\Pi_H^L = ((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes c_{H,H}) \circ ((\delta_H \circ \eta_H) \otimes H) : H \to H,$$

$$\Pi_H^R = (H \otimes (\varepsilon_H \circ \mu_H)) \circ (c_{H,H} \otimes H) \circ (H \otimes (\delta_H \circ \eta_H)) : H \to H,$$

they are idempotent and we denote by H_L, p_L and i_L the object and the morphisms such that $i_L \circ p_L = \Pi_H^L$ and $p_L \circ i_L = id_{H_L}$.

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Example.

Let G be a finite groupoid and R a commutative ring. Let G_0 be the set of objects and G_1 the set of morphisms.
The groupoid algebra is the direct product

$$RG = \bigoplus_{\sigma \in G_1} R\sigma$$

with the product of two morphisms being equal to their composition if the latter is defined and 0 in otherwise, i.e. $\sigma\tau = \sigma \circ \tau$ if $s(\sigma) = t(\tau)$ and $\sigma\tau = 0$ if $s(\sigma) \neq t(\tau)$.
The unit element is $1_{RG} = \sum_{x \in G_0} id_x$. RG is a cocommutative weak Hopf algebra, with

$$\delta_{RG}(\sigma) = \sigma \otimes \sigma, \quad \varepsilon_{RG}(\sigma) = 1, \quad \lambda_{RG}(\sigma) = \sigma^{-1}.$$

The morphisms target and source are $\Pi^L_{RG}(\sigma) = id_{t(\sigma)}, \quad \Pi^R_{RG}(\sigma) = id_{s(\sigma)}$.

Definition.

Let H be a weak Hopf algebra. We will say that A is a weak left H-module algebra if there exists a morphism $\varphi_A : H \otimes A \to A$ satisfying:

(b1) $\varphi_A \circ (\eta_H \otimes A) = id_A$.

(b2) $\varphi_A \circ (H \otimes \mu_A) = \mu_A \circ (\varphi_A \otimes \varphi_A) \circ (H \otimes c_{H,A} \otimes A) \circ (\delta_H \otimes A \otimes A)$.

(b3) $\varphi_A \circ (\mu_H \otimes \eta_A) = \varphi_A \circ (H \otimes (\varphi_A \circ (H \otimes \eta_A)))$.

(b4) $\varphi_A \circ (\Pi_H^L \otimes A) = \mu_A \circ ((\varphi_A \circ (H \otimes \eta_A) \otimes A)$.

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(b4) $\varphi_A \circ (\Pi_H^L \otimes A) = \mu_A \circ ((\varphi_A \circ (H \otimes \eta_A) \otimes A)$.

If we replace (b3) by

$$\varphi_A \circ (\mu_H \otimes A) = \varphi_A \circ (H \otimes \varphi_A)$$

we will say that (A, φ_A) is a left H-module algebra.

Let H be a weak Hopf algebra. For $n \geq 1$, we denote by H^n the n-fold tensor power $H \otimes \cdots \otimes H$. By H^0 we denote the unit object of $\mathcal{C}$, i.e. $H^0 = K$.
If $n \geq 2$, m_H^n denotes the morphism

$$m_H^n : H^n \to H$$

defined by $m_H^2 = \mu_H$ and by

$$m_H^3 = m_H^2 \circ (H \otimes \mu_H), \;\cdots\;, m_H^n = m_H^{n-1} \circ (H^{n-2} \otimes \mu_H)$$

for $k > 2$. Note that by the associativity of μ_H we have

$$m_H^n = m_H^{n-1} \circ (\mu_H \otimes H^{n-2}).$$

Let (A, φ_A) be a weak left H-module algebra and $n \geq 1$. With φ_A^n we will denote the morphism

$$\varphi_A^n : H^n \otimes A \to A$$

defined as $\varphi_A^1 = \varphi_A$ and $\varphi_A^n = \varphi_A \circ (H \otimes \varphi_A^{n-1})$. If $n > 1$, we have that

$$\varphi_A \circ (m_H^n \otimes \eta_A) = \varphi_A^{n-1} \circ (H^{n-1} \otimes (\varphi_A \circ (H \otimes \eta_A))$$

holds. In what follows, we denote the morphism $\varphi_A \circ (m_H^n \otimes \eta_A)$ by u_n and the morphism $\varphi_A \circ (H \otimes \eta_A)$ by u_1. Note that, for $n \geq 2$,

$$u_n = \varphi_A^{n-1} \circ (H^{n-1} \otimes u_1).$$

Proposition.

Let H be a weak Hopf algebra and (A, φ_A) be a weak left H-module algebra. Then, if $n \geq 1$,

$$u_n \wedge u_n = u_n.$$

Definition.

Let H be a cocommutative weak Hopf algebra and (A, φ_A) be a weak left H-module algebra. For $n \geq 1$, with

$$Reg_{\varphi_A}(H^n, A)$$

we will denote the set of morphisms $\sigma : H^n \to A$ such that there exists a morphism $\sigma^{-1} : H^n \to A$ (the inverse of σ) satisfying the following equalities:

(c1) $\sigma \wedge \sigma^{-1} = \sigma^{-1} \wedge \sigma = u_n$.

(c2) $\sigma \wedge \sigma^{-1} \wedge \sigma = \sigma$.

(c3) $\sigma^{-1} \wedge \sigma \wedge \sigma^{-1} = \sigma^{-1}$.

By $Reg_{\varphi_A}(H_L, A)$ we denote the set of morphisms $g : H_L \to A$ such that there exists a morphism $g^{-1} : H_L \to A$ (the inverse of g) satisfying

$$g \wedge g^{-1} = g^{-1} \wedge g = u_0, \;\; g \wedge g^{-1} \wedge g = g, \;\; g^{-1} \wedge g \wedge g^{-1} = g^{-1}$$

where $u_0 = u_1 \circ i_L$. Then, by (b4) of the definition of weak H-module algebra, we have

$$u_1 = u_0 \circ p_L.$$

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where $u_0 = u_1 \circ i_L$. Then, by (b4) of the definition of weak H-module algebra, we have

$$u_1 = u_0 \circ p_L.$$

Let (A, φ_A) be a weak left H-module algebra. Then, $u_0 \in Reg_{\varphi_A}(H_L, A)$, $u_n \in Reg_{\varphi_A}(H^n, A)$ and the sets $Reg_{\varphi_A}(H_L, A)$, $Reg_{\varphi_A}(H^n, A)$ are groups with neutral elements u_0 and u_n respectively. Also, if A is commutative, we have that $Reg_{\varphi_A}(H_L, A)$, $Reg_{\varphi_A}(H^n, A)$ are abelian groups.

If (A, φ_A) is a left H-module algebra, the groups $Reg_{\varphi_{\boldsymbol{A}}}(H^n, A)$, $n \geq 0$ are the objects of a semicosimplicial complex of groups with coface operators defined by

$$\partial_{0,i} : Reg_{\varphi_{\boldsymbol{A}}}(H_L, A) \to Reg_{\varphi_{\boldsymbol{A}}}(H, A), \quad i \in \{0, 1\}$$

$$\partial_{0,0}(g) = \varphi_A \circ (H \otimes (g \circ p_L \circ \Pi_H^R)) \circ \delta_H, \quad \partial_{0,1}(g) = g \circ p_L.$$

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$$\partial_{k-1,i} : Reg_{\varphi_{\mathbf{A}}}(H^{k-1}, A) \to Reg_{\varphi_{\mathbf{A}}}(H^k, A), \quad k \geq 2, \quad i \in \{0, 1, \cdots, k\}$$

$$\partial_{k-1,i}(\sigma) = \begin{cases} \varphi_A \circ (H \otimes \sigma), \quad i = 0 \\ \sigma \circ (H^{i-1} \otimes \mu_H \otimes H^{k-i-1}), \quad i \in \{1, \cdots, k-1\} \\ \sigma \circ (H^{k-2} \otimes (\mu_H \circ (H \otimes \Pi_H^L))), \quad i = k, \end{cases}$$

For this complex the codegeneracy operators are defined by

$$s_{1,0} : Reg_{\varphi_{A}}(H, A) \rightarrow Reg_{\varphi_{A}}(H_L, A),$$

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$$s_{k+1,i} : Reg_{\varphi_A}(H^{k+1},A) \to Reg_{\varphi_A}(H^k,A), \quad k \geq 1, \quad i \in \{0,1,\cdots,k\}$$

$$s_{k+1,i}(\sigma) = \sigma \circ (H^i \otimes \eta_H \otimes H^{k-i})$$

Let

$$D^k_{\varphi_A} = \partial_{k,0} \wedge \partial_{k,1}^{-1} \wedge \cdots \wedge \partial_{k,k+1}^{(-1)^{k+1}}$$

be the coboundary morphisms of the cochain complex

$$Reg_{\varphi_A}(H_L, A) \xrightarrow{D^0_{\varphi_A}} Reg_{\varphi_A}(H, A) \xrightarrow{D^1_{\varphi_A}} Reg_{\varphi_A}(H^2, A) \xrightarrow{D^2_{\varphi_A}} \cdots$$

$$\cdots \xrightarrow{D^{k-1}_{\varphi_A}} Reg_{\varphi_A}(H^k, A) \xrightarrow{D^k_{\varphi_A}} Reg_{\varphi_A}(H^{k+1}, A) \xrightarrow{D^{k+1}_{\varphi_A}} \cdots$$

associated to the cosimplicial complex $Reg_{\varphi_A}(H^\bullet, A)$.
Then, when (A, φ_A) is a commutative left H-module algebra, $(Reg_{\varphi_A}(H^\bullet, A), D^\bullet_{\varphi_A})$ gives the Sweedler cohomology of H in (A, φ_A). Therefore, the kth group, will be defined by

$$\frac{Ker(D^k_{\varphi_A})}{Im(D^{k-1}_{\varphi_A})}$$

for $k \geq 1$ and $Ker(D^0_{\varphi_A})$ for $k = 0$. We will denote it by $H^k_{\varphi_A}(H, A)$.

The normalized cochain subcomplex of $(Reg_{\varphi_{\boldsymbol{A}}}(H^{\bullet}, A), D^{\bullet}_{\varphi_{\boldsymbol{A}}})$ will be defined by

$$Reg^{+}_{\varphi_{\boldsymbol{A}}}(H^{k+1}, A) = \bigcap_{i=0}^{k} Ker(s_{k+1,i}),$$

$$Reg^{+}_{\varphi_{\boldsymbol{A}}}(H_L, A) = \{g \in Reg_{\varphi_{\boldsymbol{A}}}(H_L, A)\ ;\ g \circ p_L \circ \eta_H = \eta_A\}$$

and $D^{k+}_{\varphi_{\boldsymbol{A}}}$ the restriction of $D^{k}_{\varphi_{\boldsymbol{A}}}$ to $Reg^{+}_{\varphi_{\boldsymbol{A}}}(H^{\bullet}, A)$.
We have that $(Reg^{+}_{\varphi_{\boldsymbol{A}}}(H^{\bullet}, A), D^{\bullet +}_{\varphi_{\boldsymbol{A}}})$, is a subcomplex of $(Reg_{\varphi_{\boldsymbol{A}}}(H^{\bullet}, A), D^{\bullet}_{\varphi_{\boldsymbol{A}}})$ and the injection map induces an isomorphism of cohomology. Then,

$$H^{2}_{\varphi_{\boldsymbol{A}}}(H, A) \simeq H^{2+}_{\varphi_{\boldsymbol{A}}}(H, A) = \frac{Ker(D^{2+}_{\varphi_{\boldsymbol{A}}})}{Im(D^{1+}_{\varphi_{\boldsymbol{A}}})}.$$

Weak crossed products for weak Hopf algebras

J.M. Fernández Vilaboa, R. González Rodríguez, R., A.B. Rodríguez Raposo, Preunits and weak crossed products, J. of Pure Appl. Algebra 213 (2009), 2244-2261.

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Let A be an algebra and V be an object in $\mathcal{C}$. Suppose that there exists a morphism $\psi_V^A : V \otimes A \to A \otimes V$ such that the following equality holds

$$(\mu_A \otimes V) \circ (A \otimes \psi_V^A) \circ (\psi_V^A \otimes A) = \psi_V^A \circ (V \otimes \mu_A).$$

As a consequence, the morphism $\nabla_{A \otimes V} : A \otimes V \to A \otimes V$ defined by

$$\nabla_{A \otimes V} = (\mu_A \otimes V) \circ (A \otimes \psi_V^A) \circ (A \otimes V \otimes \eta_A)$$

is idempotent. With $A \times V$, $i_{A \otimes V} : A \times V \to A \otimes V$ and $p_{A \otimes V} : A \otimes V \to A \times V$ we denote the object, the injection and the projection associated to the factorization of $\nabla_{A \otimes V}$.

From now on we consider quadruples $\mathbb{A}_V = (A, V, \psi_V^A, \sigma_V^A)$ where A is an algebra, V an object, $\psi_V^A : V \otimes A \to A \otimes V$ a morphism satisfiying the previous identity and $\sigma_V^A : V \otimes V \to A \otimes V$ a morphism in $\mathcal{C}$.

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We say that $\mathbb{A}_V = (A, V, \psi_V^A, \sigma_V^A)$ satisfies the twisted condition if

$$(\mu_A \otimes V) \circ (A \otimes \psi_V^A) \circ (\sigma_V^A \otimes A) = (\mu_A \otimes V) \circ (A \otimes \sigma_V^A) \circ (\psi_V^A \otimes V) \circ (V \otimes \psi_V^A).$$

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$\mathbb{A}_V$ satisfies the cocycle condition if

$$(\mu_A \otimes V) \circ (A \otimes \sigma_V^A) \circ (\sigma_V^A \otimes V) = (\mu_A \otimes V) \circ (A \otimes \sigma_V^A) \circ (\psi_V^A \otimes V) \circ (V \otimes \sigma_V^A).$$

For $\mathbb{A}_V = (A, V, \psi_V^A, \sigma_V^A)$ define the product

$$\mu_{A\otimes V} = (\mu_A \otimes V) \circ (\mu_A \otimes \sigma_V^A) \circ (A \otimes \psi_V^A \otimes V)$$

and let $\mu_{A\times V}$ be the product

$$\mu_{A\times V} = p_{A\otimes V} \circ \mu_{A\otimes V} \circ (i_{A\otimes V} \otimes i_{A\otimes V}).$$

If the twisted and the cocycle conditions hold, the product $\mu_{A\otimes V}$ is associative and normalized with respect to $\nabla_{A\otimes V}$, i.e.

$$\nabla_{A\otimes V} \circ \mu_{A\otimes V} = \mu_{A\otimes V} = \mu_{A\otimes V} \circ (\nabla_{A\otimes V} \otimes \nabla_{A\otimes V}).$$

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Due to the normality condition, $\mu_{A\times V}$ is associative as well.

Definition.

If $\mathbb{A}_V = (A, V, \psi_V^A, \sigma_V^A)$ satisfies the twisted and the cocycle conditions we say that $(A \otimes V, \mu_{A\otimes V})$ is a weak crossed product.

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Definition.

If $m_{A\otimes V}$ is an associative product defined in $A \otimes V$ a preunit $\nu : K \to A \otimes V$ is a morphism satisfying

$$m_{A\otimes V} \circ (A \otimes V \otimes \nu) = m_{A\otimes V} \circ (\nu \otimes A \otimes V) = m_{A\otimes V} \circ (A \otimes H \otimes (m_{A\otimes V} \circ (\nu \otimes \nu))).$$

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Proposition.

If $(A \otimes V, \mu_{A\otimes V})$ is a weak crossed product with preunit ν, then $A \times V$ is an algebra with product $\mu_{A\times V}$ and unit $\eta_{A\times V} = p_{A\otimes V} \circ \nu$.

Examples with $\nabla_{A\otimes V} = id_{A\otimes V}$.

- Brzeziński, **Comm. in Algebra (1997)**.

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- Majid, **J. of Algebra (1994)**

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- Bespalov and Drabant, **J. of Algebra (1999)-(2001)**

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Definition.

Let H be a weak Hopf algebra, (A, φ_A) a weak left H-module algebra and $\sigma : H \otimes H \to A$ a morphism. We define the morphisms

$$\psi_H^A : H \otimes A \to A \otimes H, \quad \sigma_H^A : H \otimes H \to A \otimes H,$$

by

$$\psi_H^A = (\varphi_A \otimes H) \circ (H \otimes c_{H,A}) \circ (\delta_H \otimes A)$$

and

$$\sigma_H^A = (\sigma \otimes \mu_H) \circ \delta_{H^2}.$$

where $\delta_{H^2} = (H \otimes c_{H,H} \otimes H) \circ (\delta_H \otimes \delta_H)$

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Note that ψ_H^A satisfies

$$(\mu_A \otimes H) \circ (A \otimes \psi_H^A) \circ (\psi_H^A \otimes A) = \psi_H^A \circ (H \otimes \mu_A).$$

and then $\mathbb{A}_H = (A, H, \psi_H^A, \sigma_H^A)$ is a quadruple as in the general case.

Definition.

Let H be a cocommutative weak Hopf algebra, (A, φ_A) a weak left H-module algebra and $\sigma \in Reg_{\varphi_A}(H^2, A)$. We say that σ satisfies the twisted condition if

$$\mu_A \circ ((\varphi_A \circ (H \otimes \varphi_A)) \otimes A) \circ (H \otimes H \otimes c_{A,A}) \circ (((H \otimes H \otimes \sigma) \circ \delta_{H^2}) \otimes A)$$

$$= \mu_A \circ (A \otimes \varphi_A) \circ (\sigma_H^A \otimes A).$$

If

$$\partial_{2,3}(\sigma) \wedge \partial_{2,1}(\sigma) = \partial_{2,0}(\sigma) \wedge \partial_{2,2}(\sigma)$$

holds, we will say that σ satisfies the 2-cocycle condition.

Note that, if (A, φ_A) is a commutative left H-module algebra, the 2-cocycle condition means that $\sigma \in Ker(D^2_{\varphi_A})$.
Also, the twisted condition holds for all $\sigma \in Reg_{\varphi_A}(H^2, A)$ if A is commutative.

Proposition.

Let H be a cocommutative weak Hopf algebra, (A, φ_A) a weak left H-module algebra and $\sigma \in Reg_{\varphi_A}(H^2, A)$. The morphism σ satisfies the twisted condition if and only if $\mathbb{A}_H$ satisfies the twisted condition.

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Let H be a cocommutative weak Hopf algebra, (A, φ_A) a weak left H-module algebra and $\sigma \in Reg_{\varphi_{\mathbf{A}}}(H^2, A)$. The morphism σ satisfies the 2-cocycle condition if and only if $\mathbb{A}_H$ satisfies the cocycle condition.

Definition.

Let H be a cocommutative weak Hopf algebra, (A, φ_A) a weak left H-module algebra and $\sigma \in Reg_{\varphi_A}(H^2, A)$. We say that σ satisfies the normal condition if

$$\sigma \circ (\eta_H \otimes H) = \sigma \circ (H \otimes \eta_H) = u_1,$$

i.e., $\sigma \in Reg^{+}_{\varphi_A}(H^2, A)$.

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Proposition.

Let H be a cocommutative weak Hopf algebra, (A, φ_A) a weak left H-module algebra and $\sigma \in Reg_{\varphi_A}(H^2, A)$. Let $\mathbb{A}_H$ be the quadruple defined by φ_A, σ and assume that $\mathbb{A}_H$ satisfies the twisted and the cocycle conditions. Then, $\nu = \nabla_{A\otimes H} \circ (\eta_A \otimes \eta_H)$ is a preunit for the weak crossed product associated to $\mathbb{A}_H$ if and only if σ satisfies the normal condition.

Theorem.

Let H be a cocommutative weak Hopf algebra, (A, φ_A) a weak left H-module algebra and $\sigma \in Reg_{\varphi_A}(H^2, A)$. Let $\mathbb{A}_H$ be the quadruple defined defined by φ_A, σ and $\mu_{A\otimes H}$ the associated product. Then the following statements are equivalent:

(i) The product $\mu_{A\otimes H}$ is associative with preunit $\nu = \nabla_{A\otimes H} \circ (\eta_A \otimes \eta_H)$ and normalized with respect to $\nabla_{A\otimes H}$.

(ii) The morphism σ satisfies the twisted condition, the 2-cocycle condition and the normal condition.

From now on we will denote by $A \otimes_\tau H = (A \otimes H, \mu_{A\otimes_\tau H})$ the weak crossed product, with preunit $\nu = \nabla_{A\otimes H} \circ (\eta_A \otimes \eta_H)$, defined by $\tau \in Reg_{\varphi_A}(H^2, A)$ when it satisfies the twisted condition, the 2-cocycle condition and the normal condition. The associated algebra will be denoted by

$$A \times_\tau H = (A \times H, \eta_{A\times_\tau H}, \mu_{A\times_\tau H}).$$

Equivalent weak crossed products

Definition.

Let H be a weak Hopf algebra and (B, ρ_B) an algebra which is also a right H-comodule. The object (B, ρ_B) is called a right H-comodule algebra if the following conditions hold:

(d1) $(\mu_B \otimes \mu_H) \circ (B \otimes c_{H,B} \otimes H) \circ \circ(\rho_B \otimes \rho_B) = \rho_B \circ \mu_B$.

(d2) $(B \otimes \Pi_H^L) \circ \rho_B = ((\mu_B \circ c_{B,B}) \otimes H) \circ (B \otimes (\rho_B \circ \eta_B))$.

Theorem.

Let H be a cocommutative weak Hopf algebra, (A, φ_A) a weak left H-module algebra and $\alpha \in Reg^{+}_{\varphi_A}(H^2, A)$ such that satisfies the twisted condition and the 2-cocycle condition. Then, the algebra $A \times_\alpha H$ is a right H-comodule algebra for the coaction

$$\rho_{A \times_\alpha H} = (p_{A \otimes H} \otimes H) \circ (A \otimes \delta_H) \circ i_{A \otimes H}.$$

Definition.

Let H be a cocommutative weak Hopf algebra, (A, φ_A) a weak left H-module algebra and $\alpha, \beta \in Reg^{+}_{\varphi_A}(H^2, A)$ such that satisfy the twisted condition and the 2-cocycle condition. Let $A \otimes_{\alpha} H$, $A \otimes_{\beta} H$ the weak crossed products associated to α and β. We say that $A \otimes_{\alpha} H$, $A \otimes_{\beta} H$ are equivalent if there is an isomorphism of left A-modules and right H-comodule algebras $\omega_{\alpha,\beta} : A \times_{\alpha} H \to A \times_{\beta} H$.

Theorem.

Let H be a cocommutative weak Hopf algebra, (A, φ_A) a weak left H-module algebra and $\alpha, \beta \in Reg^{+}_{\varphi_{\mathbf{A}}}(H^2, A)$ such that satisfy the twisted condition and the 2-cocycle condition. The weak crossed products $A \otimes_\alpha H$, $A \otimes_\beta H$ associated to α and β are equivalent if and only if there exist multiplicative and preunit preserving morphisms

$$\Gamma, \Gamma' \in \ {}_A Hom^H_{\mathcal{C}}(A \otimes H, A \otimes H)$$

satisfying

$$\Gamma \circ \Gamma' = \Gamma' \circ \Gamma = \nabla_{A \otimes H},$$

$$\Gamma \circ \Gamma' \circ \Gamma = \Gamma,$$

$$\Gamma' \circ \Gamma \circ \Gamma' = \Gamma'.$$

Theorem.

Under the conditions of the previous theorem, the weak crossed products $A\otimes_{\alpha}H$, $A\otimes_{\beta}H$ associated to α and β are equivalent if and only if there exists $f \in Reg^{+}_{\varphi_{A}}(H,A)$ such that the equalities

$$(1) \quad \mu_{A}\circ(A\otimes f)\circ\psi_{H}^{A} = \mu_{A}\circ(f\otimes\varphi_{A})\circ(\delta_{H}\otimes A)$$

and

$$(2) \quad \alpha\wedge\partial_{1,1}(f) = \partial_{1,0}(f)\wedge\partial_{1,2}(f)\wedge\beta$$

hold.

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Under the conditions of the previous theorem, the weak crossed products $A\otimes_{\alpha}H$, $A\otimes_{\beta}H$ associated to α and β are equivalent if and only if there exists $f \in Reg^{+}_{\varphi_{A}}(H,A)$ such that the equalities

$$(1) \quad \mu_{A} \circ (A \otimes f) \circ \psi_{H}^{A} = \mu_{A} \circ (f \otimes \varphi_{A}) \circ (\delta_{H} \otimes A)$$

and

$$(2) \quad \alpha \wedge \partial_{1,1}(f) = \partial_{1,0}(f) \wedge \partial_{1,2}(f) \wedge \beta$$

hold.

Note that, if H is a cocommutative weak Hopf algebra, (A, φ_{A}) is a commutative weak left H-module algebra, the equality (1) is always true.
Then, if (A, φ_{A}) is a commutative left H-module algebra, the equivalence between two weak crossed products $A\otimes_{\alpha}H$, $A\otimes_{\beta}H$ is determined by the inclusion of f in $Reg^{+}_{\varphi_{A}}(H,A)$ and the equality (2). In this case (2) is equivalent to say that $\alpha \wedge \beta^{-1} \in Im(D^{1+}_{\varphi_{A}})$.

The main result

Theorem.

Let H be a cocommutative weak Hopf algebra and (A, φ_A) a commutative left H-module algebra. Then there is a bijective correspondence between $H^2_{\varphi_A}(H, A)$ and the equivalence classes of weak crossed products of $A \otimes_\alpha H$ where $\alpha : H \otimes H \to A$ satisfy the 2-cocycle condition and the normal condition.

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