Idempotent functors and classifying spaces

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Cellular spaces

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Given pointed spaces A and X, X is said **A-cellular** if it can be built from A by means of pointed homotopy colimits, possibly iterated. Moreover, a map $X \to Y$ is said to be an **A-equivalence** if it induces a weak equivalence $\mathrm{map}_*(A, X) \to \mathrm{map}_*(A, Y)$.

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Motivation

The concept of cellular space generalizes the construction of the CW-complexes using spheres as pieces (in that case $A = S^0$).

It is intended to extract information of the target space X, using as input the homotopy structure of the building space A, and taking account of the structure of the diagram (or diagrams) which build X from A.

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Cellular approximation

Given a space A, the **A-cellularization** (or A-cellular approximation) is a canonical way of turning every space into an A-cellular space which is identical to X from the point of view of A-equivalences.

A-cellularization

There exists an augmented endofunctor CW_A of the category of pointed spaces, called **A-cellularization**, such that for every space X the augmentation $CW_AX \to X$ induces a weak homotopy equivalence $\mathrm{map}_*(A, CW_AX) \simeq \mathrm{map}_*(A, X)$, and is initial among all maps $Y \to X$ which induce A-equivalence.

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This functor only makes sense in the pointed context.

Homotopy groups

The A-cellularization of a space X must be then interpreted as the closest analogue to X that can be built out of copies of A. Closeness here means in particular that the $[\Sigma^n A, CW_A X]_* \simeq [\Sigma^n A, X]_*$ for every $n \geq 0$, where ΣA stands for the suspension of A.

In the model case $A = S^0$, this amounts to say that the CW-approximation preserve the homotopy groups of the space.

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Classifying spaces

From the homotopy viewpoint, it is natural to focus in the cellularization of **classifying spaces** BG.

Recall that given a group G, BG can be defined as the G-quotient of a contractible G-space EG, and is classifying in the sense that any principal G-bundle can be obtained as a pull-back of the base with the natural map $EG \to BG$.

For any discrete G, we have that BG is connected, $\pi_1 BG = G, \pi_i BG = 0$ for $i \geq 2$. In other words, BG is an **Eilenberg**-MacLane space $K(G,1)$.

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Approximation in the primary context

Let p be a prime. We are interested in the cellularization of spaces with regard to $B\mathbb{Z}/p$, because:

- It gives information about the p-torsion of the space.
- Bring some light over the interesting and intrincate study of the mapping space $\mathrm{map}_*(B\mathbb{Z}/p, X)$, which has been the target, for example, of the **Sullivan Conjecture** and **Lannes Theory**.
- Its relationship with Group Theory.

We have investigated the $B\mathbb{Z}/p$-cellularization from two different points of view: computing its effect over different spaces, and describing its characteristics as a functor.

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Families of groups

We have studied the effect of the functor $CW_{B\mathbb{Z}/p}$ over the following families of objects:

- **Classifiying spaces of finite groups**.
- **Classifiying spaces of infinite discrete groups**.
- **Classifiying spaces of non-discrete compact Lie groups**.
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Strong closure

Before describing the values of $CW_{B\mathbb{Z}/p}$ over classifying spaces of finite groups, we need a technical definition.

Strongly closed subgroups

Let G be a finite group, $S < G$ a p-Sylow subgroup, $H < S$ a subgroup. Then H is **strongly closed** in G if whenever $h \in H$ and $g \in G$ such that $ghg^{-1} \in S$, we have that $ghg^{-1} \in H$.

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Strongly closure and completion

The concept of strong closure is crucial in the description of the cellularization of classifying spaces of finite groups, and probably some versions of it will be necessary for an analogous description of more general cases.

In the sequel the notation $X_p^\wedge$ denotes Bousfield-Kan p-completion.

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Theorem

Let G be a finite group generated by its elements of order p, let S be a Sylow p-subgroup of G, and let A be the minimal strongly closed subgroup of S containing all elements of order p in S. Then the $B\mathbb{Z}/p$-cellularization of BG has one the following shapes:

(1) If $G = S$ is a p-group then BG is $B\mathbb{Z}/p$-cellular.

(2) If G is not a p-group and $A = S$ then $CW_{B\mathbb{Z}/p}BG$ is the homotopy fiber of the natural map $BG \to \prod_{q\neq p} BG^{\wedge}_q$.

(3) If G is not a p-group and $A \neq S$ then $CW_{B\mathbb{Z}/p}BG$ is the homotopy fiber of the map $BG \to B\Gamma^{\wedge}_p \times \prod_{q\neq p} BG^{\wedge}_q$.

Here Γ is a finite non-trivial group which depends on A and its normalizer in G. In some favorable cases, $\Gamma = N_G(A)/A$.

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If G is not generated by order p elements and $\Omega_1(G)$ is the subgroup of G generated by its order p elements (which is normal), then $CW_{B\mathbb{Z}/p}B\Omega_1(G) \simeq CW_{B\mathbb{Z}/p}BG$.

It is an interesting open problem to describe the $B\mathbb{Z}/n$-cellularization of BG, when n is divided by at least two primes.

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In the context of topological compact Lie groups, it is not easy to translate the arguments of the finite case, as there is no classification of strongly closed objects, and moreover the rational structure of these groups makes impossible to use the splitting arguments needed in the previous analysis. However, we have obtained a **Serre-type dichotomy result**:

Dichotomy result

Let G be a compact connected Lie group. Then the $B\mathbb{Z}/p$-cellularization of BG is the classifying space of a p-group generated by order p elements, or else it has an infinite number of non-trivial homotopy groups.

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The case of the spheres

Now we present some concrete examples of computations of $CW_{B\mathbb{Z}/p}BG$ in this case.

- If $X = BS^1 = K(\mathbb{Z}, 2)$, it is clear comparing pointed mapping spaces that $CW_{B\mathbb{Z}/p}BS^1 = B\mathbb{Z}/p$ since $\operatorname{map}_*(B\mathbb{Z}/p, BS^1)$ is homotopically discrete with components in bijective correspondence with $\operatorname{Hom}(\mathbb{Z}/p, S^1)$.
- Consider BS^3. If $p = 2$, $CW_{B\mathbb{Z}/2}(BS^3) = B\mathbb{Z}/2$, and the augmentation is the inclusion of the center. If p is odd, $CW_{B\mathbb{Z}/p}(BS^3)^\wedge_p$ is the homotopy fiber of $BN^\wedge_p \to BK^\wedge_p$, where N is the normalizer of the maximal torus T, and K is a certain extension of the Weyl group of G.

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Orthogonal groups

- Let $O(2)$ be the orthogonal group. Then

$$CW_{B\mathbb{Z}/2}(BO(2))^{\wedge}_{2} \simeq CW_{B\mathbb{Z}/2}(BO(2)^{\wedge}_{2})^{\wedge}_{2} \simeq BO(2)^{\wedge}_{2}$$

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- The $B\mathbb{Z}/2$-cellularization of $BSO(3)$ fits in a fibration

$$(CW_{B\mathbb{Z}/2}BSO(3)) \to BSO(3)^{\wedge}_{2} \to (BSO(3)^{\wedge}_{2})_{\mathbb{Q}}.$$

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p-toral groups

We say that a discrete group P is **discrete p-toral** if it is isomorphic to an extension of a finite product of copies of $\mathbb{Z}/p^\infty$ by a finite p-group. Then:

Cellularization of classifying spaces of discrete p-toral groups

If P is discrete p-toral, and $\Omega_1(P)$ is the subgroup of P generated by the order p elements, then $CW_{B\mathbb{Z}/p}BP = B\Omega_1(P)$.

The discrete p-toral groups determine in a great part the p-torsion structure of the compact Lie groups. In particular, if T_p is an extension of a torus by a finite p-group, there is a maximal $P \subseteq T_p$ such that the natural map $CW_{B\mathbb{Z}/p}BP \to CW_{B\mathbb{Z}/p}BT_p$ is a mod p homology equivalence.

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The cellular study of classifying spaces may be generalized by changing the generation of the class, instead of the target of the functor:

Taking $A = B\mathbb{Z}/p^\infty \vee B\mathbb{Z}/p^m$ (for a certain m) instead of $B\mathbb{Z}/p$, Castellana-Gavira have proved that $CW_A(BG_p^\wedge)$ is the fibre of the rationalization of $BG_p^\wedge$, for any compact connected Lie group G. This is related with the previous result concerning $BSO(3)$.

On the other hand, any cellularization of the classifying space of a finite p-group P, and more generally a nilpotent group, is again a $K(H, 1)$. Moreover the group H is identified as the cellularization of P in the category of groups.

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Generalizations

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Composing idempotent functors

The functor CW_A is idempotent by definition, i.e. for every pointed space X, the augmentation $cw : CW_A(CW_A X) \to CW_A X$ is a homotopy equivalence, and is homotopic to the image of the augmentation $CW_A X \to X$ under the functor CW_A.

Roughly speaking, this means that CW_A does not modify A-cellular spaces.

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For the case of localizations (i.e. homotopy idempotent coaugmented functors), Rodríguez-Scevenels showed examples where the composite is not idempotent, and study the "convergence" of the series $F \circ G \circ F \circ G \dots$

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A curious counterexample

We have solved the corresponding problem, proposed by Farjoun, in which one of the functors is a localization and the other a cellularization. The main difficulty of this version comes from the fact that concrete computations of $CW_A X$ are not easily available.

Non-idempotency of the composition

Denote by CW cellularization with regard to a wedge $B\mathbb{Z}/2 \vee \bigvee_{\text{p odd prime}} M(\mathbb{Z}/p, 2)$ and denote by L a n-Postnikov section for an appropriate choice of n; then the functors LCW and CWL are not homotopy idempotent.

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We have solved the corresponding problem, proposed by Farjoun, in which one of the functors is a localization and the other a cellularization. The main difficulty of this version comes from the fact that concrete computations of $CW_A X$ are not easily available.

Non-idempotency of the composition

Denote by CW cellularization with regard to a wedge $B\mathbb{Z}/2 \vee \bigvee_{\text{p odd prime}} M(\mathbb{Z}/p, 2)$ and denote by L a n-Postnikov section for an appropriate choice of n; then the functors LCW and CWL are not homotopy idempotent.

A curious counterexample

The key idea here is to apply these functors to the classifying space of the **Suzuki simple group** $G = Sz(8)$. The hardest part is to compute explicitly $CWLCWLBG$, which differs from $CWLBG$ only in a homotopy group. It is quite likely that this process always produce spaces that are not homotopy equivalent.

Open questions

Some questions which seem natural from the explained topics:

- For G finite and not generated by order p elements, compute $CW_{B\mathbb{Z}/p}(BG_p^\wedge)$.
- Understand the role of the strongly closed objects in the cellularization of more general spaces.
- Investigate if the description of $CW_{B\mathbb{Z}/p}BSO(3)$ as a homotopy fiber of a rationalization can be generalized to other Lie groups.
- Show examples of CW and L such that their composition is not idempotent in the category of groups.

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Our results have been obtained in joint work with R. Foote and J. Scherer (finite groups), N. Castellana (discrete and compact Lie groups), and W. Chachólski, E. Farjoun and J. Scherer (nilpotent groups).

THANK YOU!!!