Recognizing mapping spaces

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Eilenberg Centennial Conference
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Joint work with

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Mapping spaces

$\mathrm{Hom}_{\mathcal{T}op}(X, Y)$ - continuous maps between topological spaces, or simplicial mapping space $\mathrm{map}(X, Y)$, play a central role in homotopy theory.
Two natural questions to ask:

(a) *Given a fixed object* $A \in \mathcal{C}$, *when is a simplicial set* X *of the form* $\mathrm{map}_*(A, Y)$ *for some* $Y \in \mathcal{C}$*?*

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Mapping out of spheres

For $A = \mathbf{S}^1$, $X = \mathrm{map}_*(\mathbf{S}^1, Y)$ has an H-group structure coming from $\nabla : \mathbf{S}^1 \to \mathbf{S}^1 \vee \mathbf{S}^1$: X is a loop space (up to w.e.) $\Leftrightarrow$ it can be rigidified to a topological group.

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Questions

What can we do for $A = \mathbf{S}^1 \vee \mathbf{S}^2$, say?
More general suspensions?
Any pointed space A?
An arbitrary object in a pointed model category $\mathcal{C}$?

Enriched Sketches & Mapping Algebras

Definition

An enriched sketch $\mathbf{\Theta}_{\mathcal{A}}$ *is a full sub-simplicial category of a pointed simplicial model category* $\mathcal{C}$ *generated from a set* $\mathcal{A}$ *under suspension and coproducts of cardinality* $< \lambda$.

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$\mathbf{\Theta}_{\mathbf{S}^n}$ *consists of all finite wedges of spheres of dimensions* $\geq n$*.*

Definition

An $\mathcal{A}$*-mapping algebra is a simplicial functor* $\mathfrak{X} : \mathbf{\Theta}_{\mathcal{A}} \to \mathcal{S}_*$ *to pointed simplicial sets* $B \mapsto \mathfrak{X}\{B\}$ *which takes coproducts to products and suspensions to loops.*

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A realizable $\mathcal{A}$*-mapping algebra* $\mathfrak{X} = \mathfrak{M}_{\mathcal{A}} Y$ *has*
$\mathfrak{X}\{B\} = \mathrm{map}_{\mathcal{C}}(B, Y) \quad (B \in \mathcal{A})$ *for a fixed* $Y \in \mathcal{C}$.

Definition

A realizable $\mathcal{A}$*-mapping algebra* $\mathfrak{M}_{\mathcal{A}} B$ *is free if* $B \in \mathcal{A}$.

Monads and their algebras

Fact

If each $B \in \mathcal{A}$ *is a homotopy cogroup object, to check w.e.'s, we only need* $\rho\mathfrak{X}\{B\} :=$

0*-simplices of path fibration* $P\mathfrak{X}\{B\} \twoheadrightarrow \mathfrak{X}\{B\}$

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The functor $\rho\mathfrak{M}_{\mathcal{A}} : \mathcal{C} \to \mathcal{M}ap^{\mathcal{A}}_{\text{red}}$ $\mathcal{M}ap^{\mathcal{A}}_{\text{red}}$ *to reduced* $\mathcal{A}$*-mapping algebras has a left adjoint* $\mathcal{L}_{\mathcal{A}} : \mathcal{M}ap^{\mathcal{A}}_{\text{red}} \to \mathcal{C}$, *defining the Stover monad* $\mathfrak{T}_{\mathcal{A}} : \mathcal{M}ap^{\mathcal{A}}_{\text{red}} \to \mathcal{M}ap^{\mathcal{A}}_{\text{red}}$.

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Fact

For any $\mathcal{A}$*-mapping algebra* $\mathfrak{X}$, $\rho\mathfrak{X}$ *is an* $\mathfrak{T}_{\mathcal{A}}$*-algebra – i.e.,* $\exists$ *map* $h : \mathfrak{T}_{\mathcal{A}}\rho\mathfrak{X} \to \rho\mathfrak{X}$ *making*

$$\begin{array}{ccc}
\mathfrak{T}_{\mathcal{A}}\mathfrak{T}_{\mathcal{A}}\rho\mathfrak{X} & \xrightarrow{\ \mu\ } & \mathfrak{T}_{\mathcal{A}}\rho\mathfrak{X} \\
{\scriptstyle \mathfrak{T}_{\mathcal{A}}h}\downarrow & & \downarrow{\scriptstyle h} \\
\mathfrak{T}_{\mathcal{A}}\rho\mathfrak{X} & \xrightarrow{\ h\ } & \rho\mathfrak{X}
\end{array}$$

Corollary

Given any $\mathcal{A}$-mapping algebra $\mathfrak{X}$, iterating $\mathfrak{T}_{\mathcal{A}}$ yields an augmented cosimplicial object $\rho\mathfrak{X} \to \mathfrak{W}^{\bullet}$ over $\mathcal{M}ap^{\mathcal{A}}_{\mathrm{red}}$, which has an extra codegeneracy in each dimension coming from the algebra structure map $h : \mathfrak{T}_{\mathcal{A}}\rho\mathfrak{X} \to \rho\mathfrak{X}$. This yields an ordinary simplicial object $V_{\bullet}$ in $\mathcal{C}$, with $[B, V_{\bullet}]_{\mathcal{C}} \to \pi_0\mathfrak{X}\{B\}$ acyclic.

Using Bousfield-Friedlander spectral sequence:

$$E^2_{s,t} = \pi_s\pi_t \operatorname{map}_*(A, V_{\bullet}) \implies \pi_{s+t} \operatorname{map}_*(A, Y)$$

we see that $Y := \|V_{\bullet}\|$ <u>might</u> realize $\mathfrak{X}$.

Problem

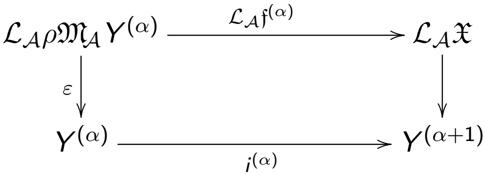
We only know that $|\operatorname{map}_*(A, V_{\bullet})\| \simeq \operatorname{map}_*(A, \|V_{\bullet}\|)$ when $A = \mathbf{S}^n$ (and each V_i is an A-CW complex).

Solution

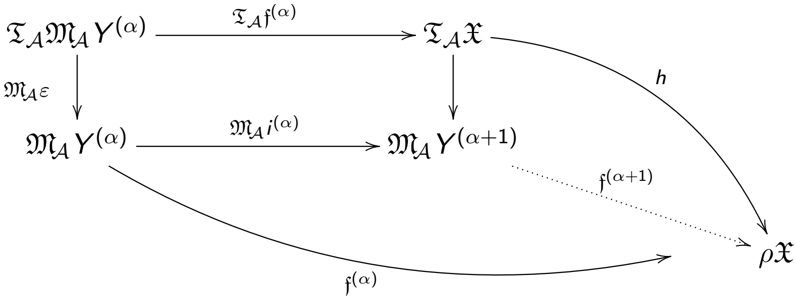
Extend the colimits used to define $\mathbf{\Theta}_{\mathcal{A}}$ by transfinite induction:

Extended mapping algebras

Let $Y^{(0)} = *$, and define $Y^{(\alpha)}$ by pushout diagram:



where $\mathfrak{f}^{(\alpha+1)}$ is defined by:



In order for the new map $\mathfrak{f}^{(\alpha+1)}$ to be determined by the inner square in the second diagram, this square must be a pushout in $\mathcal{M}ap^{\mathcal{A}}_{\mathsf{red}}$, for which we need to know that $\mathfrak{M}_{\mathcal{A}} Y^{(\alpha+1)}$ is free. Thus we have to add the pushout in the previous square to the list of limits generating $\mathbf{\Theta}_{\mathcal{A}}$ from A.

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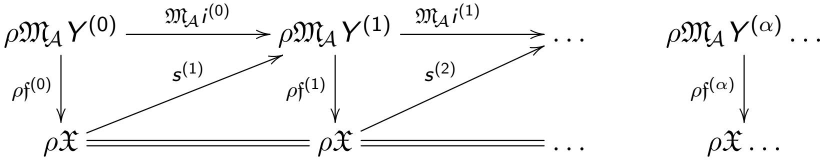
We obtain a diagram of extended $\mathcal{A}$-mapping algebras



In which the squares and lower triangles ommute (so $s^{(\alpha)}$ is a section for $\mathfrak{f}^{(\alpha)}$). This induces $s^{(\lambda)} : \rho\mathfrak{X} \to \rho\mathfrak{M}_{\mathcal{A}} Y^{(\lambda)}$, which we use to prove that $\mathfrak{f}^{(\lambda)} : \rho\mathfrak{M}_{\mathcal{A}} Y^{(\lambda)} \to \rho\mathfrak{X}$ is a weak equivalence, so $Y^{(\lambda)}$ realizes $\mathfrak{X}$.

Summary for $\operatorname{map}_*(A, -)$:

To recognize that X is w.e. to $\operatorname{map}_*(A, Y)$ in a pointed simplicial model category $\mathcal{C}$, and recover Y, we must impose an (extended) $\mathcal{A}$-mapping algebra structure on X (i.e., find $\mathfrak{X}$ with $\mathfrak{X}\{A\} \simeq X$). The amount of structure we need depends on A:

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Note

There is also a lax (homotopy-invariant) version of mapping algebras.

Dual Mapping Algebras

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A dual enriched sketch $\mathbf{\Theta}^{\mathcal{A}}$ *is a full sub-simplicial category of group objects in* $\mathcal{C}$*, closed under loops and products of* $< \lambda$ *objects.*

Example

$\mathbf{\Theta}^{\mathbb{F}_p}$ *consists of all finite* $\mathbb{F}_p$*-Eilenberg-Mac Lane spaces.*

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Failure of Eckmann-Hilton Duality

Despite the formal duality with the previous case, there are several important differences:

- In general, we will need to close $\mathcal{A}$ under *deloopings*, not only loops – so objects should be Ω^∞-spaces.

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- The mapping space $\mathrm{map}_*(Y, K(R, n))$ is a GEM (=product of Eilenberg-Mac Lane spaces), and any finite type GEM is realizable in this form.
- On the other hand, if $M = \bigoplus_{\aleph_0} \mathbb{Q}$, say, then $K(M, n)$ cannot be of the form $X = \mathrm{map}(Y, K(\mathbb{Q}, n))$ for any Y. since then

$$M = \pi_n X = H^0(Y; \mathbb{Q}) = \mathrm{Hom}_{\mathbb{Q}}(H_0(Y; \mathbb{Q}), \mathbb{Q}) .$$

Recovery of Y from $\mathrm{map}_*(Y, A)$

When $\mathcal{G}$ is a collection of Ω^∞-spaces $\mathcal{A} = (\underline{A}_n)_{n\in\mathbb{Z}}$ in $\mathcal{S}_*$, the dual Stover construction on $\rho\mathfrak{X} \in \mathcal{M}ap^{\mathrm{op}}_{\mathcal{A},\mathrm{red}}$ is defined

$$\mathcal{L}^{\mathcal{A}}(\rho\mathfrak{X}) := \prod_{\mathcal{A}\in\mathcal{G}} \prod_{n\in\mathbb{Z}} \prod_{\phi\in(\mathfrak{X}\{\underline{A}_n\})_0} Q(\phi) \ ,$$

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where $Q(\phi)$ is the pullback:

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This defines a monad $\mathcal{L}^{\mathcal{A}} \circ \rho\mathfrak{M}^{\mathcal{A}} : \mathcal{C} \to \mathcal{C}$, and $\rho\mathfrak{X}$ is a coalgebra over $\mathcal{S}_A$, so $\mathfrak{M}^{\mathcal{A}}Y$ yields a cosimplicial resolution $Y \to W^\bullet$.

Warning

This is a resolution in Bousfield's resolution model category only when $A = K(R, n)$ *for* R *a field! In general. we need to add all simplicial* R*-modules* M *to* $\mathcal{G}$*.*

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Fact

For such an $\mathcal{A}$, $\mathrm{Tot}\, W^{\bullet}$ *constructed above is the R-completion of the original Y.*

Corollary

In this case we can recover Y from $\mathfrak{X} = \mathfrak{M}^{\mathcal{A}} Y$ *(up to* $\mathcal{A}$*-w.e.)*

Recognizing $\mathrm{map}_*(Y, A)$

The above results hold only when we know that $X \simeq \mathrm{map}_*(Y, A)$ to begin with, and just want to recover Y. To *recognize* that X is such, we must restrict to the case $A = K(R, n)$ for $R = \mathbb{Q}$ or $\mathbb{F}_p$, and proceed as follows:

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The above results hold only when we know that $X \simeq \mathrm{map}_*(Y, A)$ to begin with, and just want to recover Y. To *recognize* that X is such, we must restrict to the case $A = K(R, n)$ for $R = \mathbb{Q}$ or $\mathbb{F}_p$, and proceed as follows:

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- Show that $Y := \mathrm{Tot}\, W^\bullet$ realizes $\mathfrak{X}$ by using Bousfield's *homology* spectral sequence for $W^\bullet$.

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Given an Ω^∞-space $A \in \mathcal{S}_*$, to recoover Y from $X = \mathrm{map}_*(Y, A)$, we must impose a dual $\mathcal{G}$-mapping algebra structure on X for some collection of Ω^∞-spaces $\mathcal{G}$ depending on X and Y:

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Note

when $A = K(R, n)$ *for* $R = \mathbb{Q}$ *or* $\mathbb{F}_p$, *we also have recognition principle to tell if an R-GEM X is w.e to* $\mathrm{map}_*(Y, A)$ *for some simply connected space Y – if we can impose a dual R-mapping algebra structure on X.*

Why do we care?

The question of recovering Y from $\mathrm{map}_*(Y, K(R, n))$ seems rather esoteric (though it is a little surprsing that one can recover Y from an R-GEM).

Instead, think of the three notions associated to a solid ring R:

(a) A dual $\mathbf{\Theta}^R$-dual mapping algebra $\mathfrak{X}$.
(b) a cosimplicial weak $\mathcal{G}$-resolution $W^\bullet$ realizing a simplicial Θ^R-resolution of $\Lambda := \pi_0\mathfrak{X}$.
(c) The R-complete space $Y = \mathrm{Tot}\, W^\bullet$.

For $R = \mathbb{F}_p$ or $\mathbb{Q}$, under mild assumptions they are equivalent. Moreover, by choosing $V_\bullet \to \Lambda$ carefully,we can describe explicitly the additional data needed to determine $W^\bullet$ in terms of *higher cohomology operations*.