Model structures for Goodwillie's calculus of homotopy functors

Georg Biedermann

Universität Osnabrück
Joint work with Oliver Röndigs

Eilenberg**100**, Warszawa

25.7.2013

Notation

Let us denote by

$\mathcal{S}$ the category of spaces (simplicial sets)

$\mathcal{S}_*$ pointed spaces

Sp (Bousfield-Friedlander) spectra

Let $\mathcal{C}$ and $\mathcal{D}$ be any of these categories.

A functor from $\mathcal{C}$ to $\mathcal{D}$ is a *homotopy functor* if it sends weak equivalences to weak equivalences.

Taylor towers

Theorem (Goodwillie)

For every homotopy functor $F\colon \mathcal{C} \to \mathcal{D}$ there exists a tower

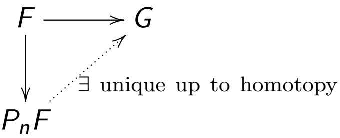
$$F \to \ldots \to P_nF \to \ldots P_1F \to P_0F$$

of homotopy functors under F such that the map

$$F \to P_nF$$

is universal up to objectwise weak equivalence among all maps from F to an n-excisive functor.

"Universal" means: Let G be n-excisive.



Cubical diagrams (Cubes)

Let $\underline{n} = \{1, \ldots, n\}$ and $P(\underline{n})$ the power set. A functor

$$K\colon\ P(\underline{n}) \to \mathcal{C}$$

is an *n-cube* in $\mathcal{C}$.

An n-cube is

Cartesian if the map $K_\emptyset \to \mathrm{holim}_{\emptyset \neq S \subset \underline{n}} K_S$ is a weak equivalence.

strongly coCartesian if every 2-dimensional face is a homotopy pushout square.

Higher Excision

Definition

A homotopy functor is n-excisive if it sends all strongly coCartesian $(n+1)$-cubes to Cartesian ones.

For small n:

- F 0-excisive means: $F(K) \simeq F(*)$ for all K, i.e. F is homotopy constant
- F 1-excisive means: F sends homotopy pushouts to homotopy pullbacks

1-excisive = gen. homology theory

$$\begin{array}{ccc} A\cap B & \longrightarrow & A \\ \downarrow & & \downarrow \\ B & \longrightarrow & A\cup B \end{array} \quad \rightsquigarrow \quad \begin{array}{ccc} F(A\cap B) & \longrightarrow & F(A) \\ \downarrow & & \downarrow \\ F(B) & \longrightarrow & F(A\cup B) \end{array}$$

Applying π_* yields a Mayer-Vietoris sequence

$$\begin{aligned} \ldots \to \pi_* F(A\cap B) &\to \pi_* F(A) \oplus \pi_* F(B) \to \pi_* F(A\cup B) \\ &\to \pi_{*-1} F(A\cap B) \to \ldots \end{aligned}$$

F (reduced) 1-excisive $\Leftrightarrow$ $\pi_* F$ is a (reduced) gen. homology theory.

From stable to unstable

Most interesting example:

$$F = \mathrm{id}\colon \mathcal{T}_* \to \mathcal{T}_*$$

It gives rise to π_*, the unstable homotopy groups.

Its linearization is stable homotopy theory:

$$P_1 \mathrm{id}_{\mathcal{T}_*} = \Omega^\infty \Sigma^\infty$$

For a general reduced functor F we have

$$P_1 F = \mathrm{hocolim}_k \, \Omega^k F \Sigma^k = \Omega^\infty F \Sigma^\infty.$$

Homogeneous functors

From now on: reduced functors from $\mathcal{T}_*$ to $\mathcal{T}_*$ or Sp.

More definitions:

F is *reduced* if
$$F(*) \simeq *$$

F is *n-reduced* if
$$P_{n-1}F \simeq *$$

F is *n-homogeneous* $\Leftrightarrow$ F is n-excisive and n-reduced

1-homogeneous $=$ linear

The classification of homogeneous functors

We can form the *n-homogeneous part* of F by

$$D_n F = \mathrm{hofib}[P_n F \to P_{n-1} F]$$

Theorem (Goodwillie)

For any homotopy functor $F\colon \mathcal{S}_*, \mathrm{Sp} \to \mathcal{S}_*, \mathrm{Sp}$ there exists a Σ_n-spectrum $\partial^n F$ and an objectwise weak equivalence

$$D_n F(X) \simeq \Omega^\infty(\partial^n F \wedge X^{\wedge n})_{h\Sigma_n}.$$

Compare with the degree n part of the Taylor series of f:

$$\frac{f^{(n)}(0) \cdot x^n}{n!}.$$

In fact, Goodwillie proves that there is a commuting diagram of adjoint equivalences:

$$\begin{array}{ccc}
\mathrm{Sp}^{\Sigma_n} & & \\
\mathcal{L}\mathrm{Ev} \downarrow \;\; \uparrow \mathrm{Ev} & & \\
\mathrm{Ho}(\text{sym. mlin. fun. } (\mathcal{T}_*)^n \to \mathrm{Sp}) & \underset{\mathrm{hocr}_n}{\overset{\mathcal{L}_n}{\rightleftarrows}} & \mathrm{Ho}(n\text{−hom. functors } \mathcal{T}_* \to \mathrm{Sp}) \\
\Sigma^\infty \uparrow \;\; \downarrow \Omega^\infty & & \Sigma^\infty \uparrow \;\; \downarrow \Omega^\infty \\
\mathrm{Ho}(\text{sym. mlin. fun } (\mathcal{T}_*)^n \to \mathcal{T}_*) & \underset{\mathrm{hocr}_n}{\overset{\mathcal{L}_n}{\rightleftarrows}} & \mathrm{Ho}(n\text{−hom. functors } \mathcal{T}_* \to \mathcal{T}_*)
\end{array}$$

$$\begin{array}{ccc}
E \wedge K_1 \wedge \ldots \wedge K_n & \mapsto & (E \wedge K \wedge \ldots \wedge K)_{h\Sigma_n} \\
P_{1,\ldots,1}\mathrm{hocr}_n F(K_1, \ldots, K_n) & \leftarrow\!\shortmid & F
\end{array}$$

Cross effects

Let $F\colon \mathcal{T}_* \to \mathcal{T}_*, \mathrm{Sp}$. Then the *n-th homotopy cross effect* is

$$\mathrm{hocr}_n F(K_1, \ldots, K_n) = \mathrm{hofib}\left[F(\bigvee_{i=1}^{n} K_i) \to \mathrm{holim}_{\emptyset \neq S \subset \underline{n}} F(\bigvee_{i \notin S} K_i)\right]$$

We also will consider the *(strict) cross effect*

$$\mathrm{cr}_n F(K_1, \ldots, K_n) = \mathrm{fib}\left[F(\bigvee_{i=1}^{n} K_i) \to \lim_{\emptyset \neq S \subset \underline{n}} F(\bigvee_{i \notin S} K_i)\right]$$

Our setup: let $\mathcal{C}$ and $\mathcal{D}$ be categories satisfying some technical conditions, eg

- $\mathcal{D}$ is a pointed cofibrantly generated proper $\mathcal{S}_*$-model category, directed hocolims commute with finite holims in $\mathcal{D}$
- $\mathcal{C}$ small full subcategory of a $\mathcal{S}_*$-model category, has only cofibrant and finitely presentable objects
- $\mathcal{C}$ closed under join with finite sets

Consider $\mathcal{S}_*$-functors $\mathrm{Fun}_{\mathcal{S}_*}(\mathcal{C}, \mathcal{D})_{\mathrm{proj}}$ with the projective model structure.

Left Bousfield localizations

Perform successive left Bousfield localizations:

$$\mathrm{Fun}_{\mathcal{S}_*}(\mathcal{C},\mathcal{D})_{\mathrm{proj}} \to \mathrm{Fun}_{\mathcal{S}_*}(\mathcal{C},\mathcal{D})_{\mathrm{hf}} \to \mathrm{Fun}_{\mathcal{S}_*}(\mathcal{C},\mathcal{D})_{n-\mathrm{exc}}$$

(done before by Dwyer, B/Chorny/Röndigs, Perreira)

The wreath product, the cr model structure

Symmetry of functors can be expressed in the source category:

$$\Sigma_n \wr \mathcal{C}^{\wedge n}$$

We want $\mathrm{cr}_n \colon \mathrm{Fun}(\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}(\Sigma_n \wr \mathcal{C}^{\wedge n}, \mathcal{D})$ to be a right Quillen functor such that $R\mathrm{cr}_n \simeq \mathrm{hocr}_n$.

There is such a model structure on $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ called the cr model structure. So:

$$\mathrm{Fun}_{\mathcal{S}_*}(\mathcal{C}, \mathcal{D})_{\mathrm{cr}} \rightsquigarrow \mathrm{Fun}_{\mathcal{S}_*}(\mathcal{C}, \mathcal{D})_{\mathrm{hf-cr}} \rightsquigarrow \mathrm{Fun}_{\mathcal{S}_*}(\mathcal{C}, \mathcal{D})_{n-\mathrm{exc-cr}}$$

Right Bousfield localization

Observe:

$$\mathrm{cr}_n F(K_1, \ldots, K_n) = \mathrm{map}(\bigwedge_{i=1}^{n} R^{K_i}, F)$$

Perform right Bousfield localization with respect to the set:

$$\left\{ \bigwedge_{i=1}^{n} R^{K_i} \mid K_1, \ldots, K_n \in \mathcal{D} \right\}$$

We obtain:

$$\mathrm{Fun}_{\mathcal{S}_*}(\mathcal{C}, \mathcal{D})_{n-\mathrm{exc}-\mathrm{cr}} \rightsquigarrow \mathrm{Fun}_{\mathcal{S}_*}(\mathcal{C}, \mathcal{D})_{n-\mathrm{hom}}$$

Exists by a theorem of Christensen/Isaksen.

The Quillen equivalences

We obtain a commuting diagram of $\mathcal{S}_*$-Quillen equivalences inducing Goodwillie's diagram:

$$\begin{array}{ccc}
\mathrm{Fun}(\Sigma_n \wr \mathcal{C}^{\wedge n}, \mathrm{Sp}(\mathcal{D}))_{\mathrm{ml}} & \underset{\mathrm{cr}_n}{\overset{\mathcal{L}_n}{\rightleftarrows}} & \mathrm{Fun}(\mathcal{C}, \mathrm{Sp}(\mathcal{D}))_{n-\mathrm{hom}} \\
\Sigma^\infty \uparrow \downarrow \Omega^\infty & & \Sigma^\infty \uparrow \downarrow \Omega^\infty \\
\mathrm{Fun}(\Sigma_n \wr \mathcal{C}^{\wedge n}, \mathcal{D})_{\mathrm{ml}} & \underset{\mathrm{cr}_n}{\overset{\mathcal{L}_n}{\rightleftarrows}} & \mathrm{Fun}(\mathcal{C}, \mathcal{D})_{n-\mathrm{hom}}
\end{array}$$

There is an $\mathcal{S}_*$-Quillen equivalence:

$$\mathrm{Sp}(\mathcal{D})^{\Sigma_n} \rightleftarrows \mathrm{Fun}(\Sigma_n \wr (\mathcal{S}_*^{\mathrm{fin}})^{\wedge n}, \mathrm{Sp}(\mathcal{D}))_{\mathrm{ml}}$$